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CONTENTS

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A Resolution of Normal Surfaces of Triple Section Type

TADASHI ASHIKAGA

Introduction. In this note, "a surface" means an algebraic surface defined over the complex number field. In [5, §2], Horikawa introduced a method for resolving singularities on normal surfaces of double section type in the completion of the total space of a line bundle over a surface. This method is sometimes useful not only for studying surfaces globally ([6], [13], [14] etc.) but also for studying isolated singularities locally ([17] etc.).

In fact, let (V, P) be a normal 2-dimentional double point. Since the embedded dimension of (V, P) is 3 by [1], an analytic equation of P is given by $\xi^2 + f(x, y) = 0$ from a suitable change of the coordinate. Then one can obtain a resolution of P from a resolution of the plane curve singularity $f(x, y) = 0$ by this method. The origin of this idea is classical ([4], [7], [10],etc.), but the essential point of [5, §2] is to obtain a formula for analytic invariants of P at the same time as resolving P ([5, p.50]).

Now in this note, we extend this method to normal surfaces of triple section type. (A special form of this is already seen in [2, §3].) The local version of our method is the following : Let (V, P) be a normal 2-dimentional hypersurface triple point. By a suitable change of the coordinate, the equation of P is given by $\xi^3 + g(x, y)\xi + h(x, y) = 0$. Then, from the pair of the resolution of plane curve singularities $g(x, y) = 0$ and $h(x, y) = 0$, one can descend the multiplicity of any infinitely near point of P ; that is, one can reduce this triple point to some isolated double points and simple codimension 1 double points. (We call them compound nodes and compound cusps.)

In §1, we produce the above process by twisting the line bundles similar as in [5, §2], and obtain a certain formula. In §2, we give some examples. In §3, we show the following : Let (V, P) be an isolated singularity defined by $\xi^3 + f(x, y) = 0$. Let μ be the Milnor number of P, and let p_g be the geometric genus of P. Then we have $\mu \geq 6p_g$. Durfee's conjecture [3, p.97] is true in this case. (Compare [20] and [19].)

Acknowledgement I express my hearty thanks to Prof.Kazuhiro Konno for many helpful comments during the seminars we held, to Prof.Masataka Tomari for very useful advices.

1 Process of resolution.

1.1 Let Y be a nonsingular surface, and let L be a line bundle on Y. Let $\pi : X = P(\mathcal{O}_Y(L) \oplus \mathcal{O}_Y) \longrightarrow Y$ be a $\mathbb{P}^1$-bundle, and set $T = \mathcal{O}_X(1)$ the 0-section of π and set T_∞ the ∞-section of π. T_∞ is linearly equivalent to $T - \pi^*L$. Let S be a normal surface on X which is linearly equivalent to $3T$. We choose $X_0 \in |T|$ (the complete linear system of T) and $X_1 \in |T_\infty|$ such that (X_0, X_1) gives a system of homogeneous fiber coordinate of π. Then the equation of S is given by $\Sigma_{i=0}^3 \psi_{iL} X_0^{3-i} X_1^i = 0$ where ψ_{iL} is an element of $H^0(Y, \mathcal{O}(iL))$. By setting $Y_0 = X_0 + (1/3)\psi_L X_1 \in |T|, Y_1 = X_1, g = \psi_{2L} - (1/3)\psi_L^2 \in H^0(Y, \mathcal{O}(2L))$ and $h = \psi_{3L} - (1/3)\psi_{2L}\psi_L + (2/27)\psi_L^3 \in H^0(Y, \mathcal{O}(3L))$, the equation of S is given by $Y_0^3 + gY_0Y_1^2 + hY_0^3 = 0$. The surface S does not intersect with T_∞. So for convenience, setting $\xi = Y_0/Y_1$ the inhomogeneous fiber coordinate of π, we write the equation of S by

$$f := \xi^3 + g\xi + h = 0. \tag{1}$$

We call the divisors $G = (g)$ and $H = (h)$ on Y *the first and the second assistant curve* for S respectively.

Let $\widetilde{P}$ be a singular point on S. Set $P = \pi(\widetilde{P})$. Let m_P be the maximal ideal at P, and let $\Delta = 4g^3 + 27h^2 \sim 6L$ be the discriminant divisor for f. ($\sim$ is the linear equivalence.) By [12, Lemma 5.1], $\widetilde{P}$ satisfies one of the following two conditions ;

(a) $g \in m_P$ and $h \in m_P^2$ or,

(b) $g \notin m_P, h \notin m_P$ and $\Delta \in m_P^2$.

When (b) is the case, set $\widetilde{P} = (\xi, x, y) = (\xi_0, x_0, y_0)$ for $\xi_0 \neq 0$ where (x, y) is a local coordinate at P. Then by putting $\eta = \xi - \xi_0$, we have

$$f = \eta^3 + 3\xi_0\eta^2 + (g + 3\xi_0^2)\eta + (\xi_0^3 + g\xi_0 + h),$$

where $g(x_0, y_0) + 3\xi_0^2 = 0, g_x(x_0, y_0)\xi_0 + h_x(x_0, y_0) = 0$ and $g_y(x_0, y_0) + h_y(x_0, y_0) = 0$. Hence $\widetilde{P}$ is a double point. We call it *an inner double point* . When (a) is the case, $\widetilde{P}$ is on the 0-section $\xi = 0$. We call it *a target singularity.* From now on, we consider to reduce the target singularities of multiplicity 3 on S to some isolated double points and simple codimension one double points.

1.2 Let $\widetilde{P}$ be a target triple point on S. Assume $\widetilde{P} = \{(\xi, x, y) = (0, 0, 0)\}$ and the equation at $\widetilde{P}$ is of the form (1). Set $m_1 = mult_P G \geq 2$ and $n_1 = mult_P H \geq 3$ where $mult_P G$ is the multiplicity of G at P. Let $\tau_1 : Y_1 \longrightarrow Y$ be the blow up at P, and set $E_1 = \tau^{-1}(P)$. Let $\widetilde{G_1}, \widetilde{H_1}$ be the proper transform of G and H by τ_1 respectively. Then we have $\tau_1^* G = \widetilde{G_1} + m_1 E_1$ and $\tau_1^* H = \widetilde{H_1} + n_1 E_1$. Now we set

$$l_1 = min([m_1/2], [n_1/3]),$$

where $[m_1/2]$ is the greatest integer not exceeding $m_1/2$. Put $L_1 = \tau_1^* L - l_1 E_1$. Let $\pi_1 : X_1 = \mathbf{P}(\mathcal{O}_{Y_1}(L_1) \oplus \mathcal{O}_{Y_1}) \longrightarrow Y_1$ be the $\mathbf{P}^1$ -bundle, and set $T_1 = \mathcal{O}_{X_1}(1)$. Put $G_1 = \tau_1^* G - 2l_1 E_1 = \widetilde{G_1} + (m_1 - 2l_1)E_1 \sim 2L_1$ and $H_1 = \tau_1^* H - 3l_1 E_1 = \widetilde{H_1} + (n_1 - 3l_1)E_1 \sim 3L_1$. We have either $0 \leq m_1 - 2l_1 \leq 1$ or $0 \leq n_1 - 3l_1 \leq 2$. Now we set

$$f_1 = \xi_1^3 + g_1\xi + h_1,$$

where ξ_1 is the inhomogeneous fiber coordinate of π_1, $(g_1) = G_1$ and $(h_1) = H_1$. The surface S_1 on X_1 defined by (f_1) is linearly equivalent to $3T_1$. Let $X_1 \longrightarrow X$ be the morphism associated with the composition of sheaf homomorphisms

$$\mathcal{O}_Y(L) \xrightarrow{\tau_1^*} \mathcal{O}_{Y_1}(\tau_1^* L) \longrightarrow \mathcal{O}_{Y_1}(L_1),$$

where the last map is obtained by tensoring $\mathcal{O}_{Y_1}(-l_1 E_1)$. Let $\widetilde{\tau_1} : S_1 \longrightarrow S$ be the restriction of this morphism to S_1. We complete the first step of the following commutative diagram ;

$$
\begin{array}{lccccc}
 & X & & X_1 & & X_r \\
 & \cup & & \cup & & \cup \\
(2) & S & \xleftarrow{\tilde{\tau}_1} & S_1 & \leftarrow \cdots \xleftarrow{\tilde{\tau}_n} & S_r \\
 & \downarrow \pi & & \downarrow \pi_1 & & \downarrow \pi_r \\
 & Y & \xleftarrow{\tau_1} & Y_1 & \leftarrow \cdots \xleftarrow{\tau_n} & Y_r .
\end{array}
$$

We continue this process. Then :

Lemma 1.3 *After a finite step of this process (say r-times), the obtained surface S_r has no singularity of multiplicity 3.*

In the diagram (2), we assume that $\tau_i : Y_i \longrightarrow Y_{i-1}$ is the blow up at $P_{i-1} \in Y_{i-1}$, and set $m_i = mult_{P_{i-1}} G_{i-1}, n_i = mult_{P_{i-1}} H_{i-1}$ and $l_i = min([m_i/2], [n_i/3])$. Let ω_{S_i} be the dualizing sheaf on S_i.

Lemma 1.4

$$
\begin{aligned}
&\chi(\mathcal{O}_{S_r}) - \chi(\mathcal{O}_S) = -(1/2)\textstyle\sum_{i=1}^{r} l_i(5l_i - 3), \\
&\omega_{S_r}^2 - \omega_S^2 = -3\textstyle\sum_{i=1}^{r}(2l_i - 1)^2.
\end{aligned}
$$

1.5 We study the assistant curves of S_r. We may assume that the reduced part of G_r and H_r is normal crossing. Let $\mathbb{E} = \sum_{i=1}^{k} E_i$ be the decomposition to the irreducible components of the total transform of the curve $G \cup H$ by $\tau = \tau_r \circ \cdots \circ \tau_1$. Assume that $\mathbb{E}' = \sum_{i=1}^{r} E_i$ is the exceptional curve for τ. For each $E_i \in \mathbb{E}$, we give a double $\mathbb{Z}$-weighting $(a_i : b_i)$ such that E_i is the component of G_r (resp. H_r) of multiplicity a_i (resp. b_i). By this way, the components of $\mathbb{E}$ are classified into the following six types ;

$$
\begin{array}{ll}
(C)\ (\alpha, 2)\ ,\ \alpha \geq 2 & (N)\ (1, \beta)\ ,\ \beta \geq 2 \\
(I)\ (\gamma, 1)\ ,\ \gamma \geq 1 & (II)\ (\delta, 0)\ ,\ \delta \geq 1 \\
(III)\ (0, \epsilon)\ ,\ \epsilon \geq 1 & (\phi)\ (0, 0).
\end{array}
$$

Moreover, continue our process (2) if necessary. Then we can classify the type of points of intersection of the components of $\mathbb{E}$ in the following ;

$$
\begin{array}{llll}
P(C\cap II), & P(N^2\cap II), & P(N\cap III), & P(I\cap II),\\
P(I\cap III^1), & P(I^1\cap III), & P(II\cap II), & P(II\cap III^2),\\
P(II\cap III^1), & P(II^1\cap III), & P(III\cap III), & P(\phi\cap \mathit{any\,type}),
\end{array}
$$

where $P(C\cap II)$ is the point of intersection of the components of types (C) and (II), $P(N^2\cap II)$ is that of types (N) with $\beta=2$ and (II), etc.

1.6 We proceed our resolution process.

1) Let C_i $(1\leq i\leq n(C))$ be the component of $\mathbf{E}'$ of type (C). Set $\widetilde{C_i}=\pi_r^{-1}(C_i)$. The singularity of S_r is $\xi^3+x^\alpha\xi+x^2=0$ at a general point of $\widetilde{C_i}$, and is $\xi^3+x^\alpha y^\delta\xi+x^2=0$ at the point of type $\pi_r^{-1}(P(C_i\cap II))$. We call it a *compound cusp.*

Let $\tau_{r+1}: X_{r+1}\longrightarrow X_r$ be the blow up with the center $\Sigma_{i=1}^{n(C)}\widetilde{C_i}$. Let S_{r+1} be the strict transform of S_r by τ_{r+1}. Set $\bar\tau_{r+1}=\tau_{r+1}|_{S_{r+1}}$ be the restriction of τ_{r+1} to S_{r+1}. The surface S_{r+1} is nonsingular along $(\bar\tau_{r+1})^{-1}(\widetilde{C_i}), 1\leq i\leq n(C)$. Let C_i' be the reduced part of $(\bar\tau_{r+1})^{-1}(\widetilde{C_i})$. C_i' is naturally isomorphic to $\mathbf{P}^1$. And we have

$$
\begin{aligned}
&\chi(\mathcal{O}_{S_{r+1}})-\chi(\mathcal{O}_{S_r})=n(C)-\Sigma_{i=1}^{n(C)}(C_i')^2,\\
&\omega_{S_{r+1}}^2-\omega_{S_r}^2=8n(C).
\end{aligned}
$$

2) Let $N_i (1\leq i\leq n(N))$ be the component of $\mathbf{E}'$ of type (N). Assume that N_i has $\mathbf{Z}$-weighting $(1,2)$ (resp. $(1,\beta)$ with $\beta\geq 3$) for $1\leq i\leq n(N^2)$ (resp. $n(N^2)+1\leq i\leq n(N)$). Set $\tau_{r+1}'=\bar\tau_{r+1}\circ\pi_r$ and $\widetilde{N_i}=(\tau_{r+1}')^{-1}(N_i)$. The singularity along $\widetilde{N_i}$ is the following ;

$$
\begin{aligned}
&\xi^3+x\xi+x^\beta=0 \text{ at a general point of } \widetilde{N_i},\\
&\xi^3+x\xi+x^\beta y^\epsilon=0 \ \text{ at } (\tau_{r+1}')^{-1}(P(N_i\cap III)) \ \text{ for } 1\leq i\leq n(N),\\
&\xi^3+xy^\delta\xi+x^2=0 \ \text{ at } (\tau_{r+1}')^{-1}(P(N_i\cap II)) \ \text{ for } 1\leq i\leq n(N^2).
\end{aligned}
$$

We call it a *compound node.*

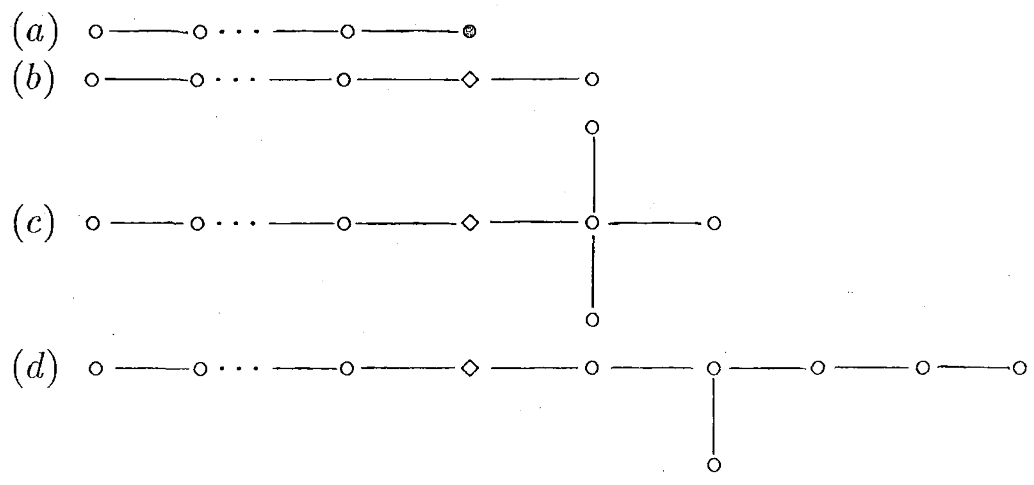
Let $\tau_{r+2}: X_{r+2}\longrightarrow X_{r+1}$ be the blow up with the center $\Sigma_{i=1}^{n(N)}\widetilde{N_i}$. Let S_{r+2} be the strict transform of S_{r+1} by τ_{r+2}. We set $\bar\tau_{r+2}=\tau_{r+2}|_{S_{r+2}}$ and $\tau_{r+2}'=\bar\tau_{r+2}\circ\tau_{r+1}'$. The pull back $(\bar\tau_{r+2})^*(\widetilde{N_i})$ is a union with two nonsingular rational curves N_i' and N_i''. The surface $S_{\tau+2}$ is nonsingular along N_i' and N_i''. Moreover, for $n(N^2) < i \leq n(N)$, N_i' and N_i'' does not intersect each other. For $1 \leq i \leq n(N^2)$, we let $II_1, \cdots, II_{n(N_i II)}$ be the components of $\mathbb{E}$ of type (II) which intersect N_i. Assume that the $\mathbb{Z}$-weighting of II_j is $(\delta_j, 0)$ for $1 \leq j \leq n(N_i II)$. Then the curves N_i' and N_i'' intersect each other at $(\tau'_{\tau+2})^{-1}(P(N_i \cap II_j))$ with intersection multiplicity δ_j. And we have

$$\begin{aligned} \chi(\mathcal{O}_{S_{\tau+2}}) - \chi(\mathcal{O}_{S_{\tau+1}}) &= n(N) - \textstyle\sum_{i=1}^{n(N^2 II)} \sum_{j=1}^{n(N_i II)} \delta_j, \\ \omega^2_{S_{\tau+2}} - \omega^2_{S_{\tau+1}} &= 8n(N) + \textstyle\sum_{i=1}^{n(N)}((N_i')^2 + (N_i'')^2) \\ &\quad -2 \textstyle\sum_{i=1}^{n(N^2 II)} \sum_{j=1}^{n(N_i II)} \delta_j. \end{aligned}$$

3) At the point of $(\tau'_{\tau+2})^{-1}(P(I^\tau \cap III^\epsilon))$, the surface $S_{\tau+2}$ has a rational double point (a RDP for short) of type $A_{3\epsilon-1}$. Put $\bar{\tau}_{\tau+3} : S_{\tau+3} \longrightarrow S_{\tau+2}$ the minimal resolution of these singularities. We have

$$\chi(\mathcal{O}_{S_{\tau+3}}) = \chi(\mathcal{O}_{S_{\tau+2}}), \; \omega^2_{S_{\tau+3}} = \omega^2_{S_{\tau+2}}.$$

4) At the point of $(\bar{\tau}_{\tau+3} \circ \tau'_{\tau+2})^{-1}(P(II^\delta \cap III^2))$, the surface $S_{\tau+3}$ has a singularity $\xi^3 + x^\delta \xi + y^2 = 0$. Let $\bar{\tau}_{\tau+4} : S_{\tau+4} \longrightarrow S_{\tau+3}$ be the minimal resolution. If $\delta \leq 3$, then this is a RDP. If $\delta \geq 4$, then the dual graph of the exceptional curve of this singularity is the following :



In the above graph, the symbol $\circ$ (resp. $\diamond$) is a nonsingular rational curve with self-intersection number -2 (resp. -3). The symbol

$\circledast$ is an elliptic curve with self-intersection number -3. The graphs $(a),(b),(c),(d)$ correspond to $\delta\equiv 0,1,2,3 \pmod 4$, respectively. The number of (- 2)-curves of the head part $\circ - - \circ\cdots\circ$ in $(a)\sim(d)$ is $[\delta/4]-1$. Moreover we have

$$\chi(\mathcal{O}_{S_{r+4}})-\chi(\mathcal{O}_{S_{r+3}})=\omega^2_{S_{r+4}}-\omega^2_{S_{r+3}}=\sum_{i=1}^{n(III^2II)}[\delta_i/4],$$

where the summation moves all the points of type $P(III^2\cap II^{\delta_i})$.

We set $\bar\tau=\bar\tau_{r+4}\circ\cdots\circ\bar\tau_1: S^*=S_{r+4}\longrightarrow S$, and call it the *canonical resolution* of S.

Let ϕ_i be a component of **E**' of type (ϕ). Let $E_1,\cdots,E_{l(i)}$ be the components of **E** which intersect to ϕ_i. Assume that E_j has **Z**-weighting (a_j,b_j) for $1\le j\le l(i)$. Then we have $3\sum_{i=1}^{l(i)}a_j=2\sum_{i=1}^{l(i)}b_j$. Let Δ_r be the strict transform of the discriminant divisor Δ by τ. If Δ_r intersect ϕ_i at a point such that $\Delta_r\in m_P^2$, there is a possibility that S^* is singular at $(\bar\tau_{r+4}\circ\cdots\circ\bar\tau_{r+1}\circ\pi_r)^{-1}(P)$. We call it an *infinitely near inner double point.*

Proposition 1.7 *The surface S^* has only inner double and infinitely near inner double points as its singularities (if they exists). Moreover we have*

$$\begin{aligned}\chi(\mathcal{O}_{S^*})-\chi(\mathcal{O}_S)&=-(1/2)\textstyle\sum_{i=1}^{r}l_i(5l_i-3)+n(C)+n(N)\\&\quad-\textstyle\sum_{i=1}^{n(C)}(C_i')^2-\sum_{i=1}^{n(N^2II)}\delta_i-\sum_{i=1}^{n(III^2II)}[\delta_i/4],\\ \omega^2_{S^*}-\omega^2_S&=-3\textstyle\sum_{i=1}^{r}(2l_i-1)^2+8n(C)+8n(N)\\&\quad+\textstyle\sum_{i=1}^{n(N)}((N_i')^2+(N_i'')^2)-2\sum_{i=1}^{n(N^2II)}\delta_i-\sum_{i=1}^{n(III^2II)}[\delta_i/4].\end{aligned}$$

2 Examples.

Let (V,P) be a 2-dimensional hypersurface singularity of multiplicity 3. By the Weierstrass preparation theorem and the Tschirnhauss-transformation, the equation of (V,P) is given by $\xi^3+g(x,y)\xi+h(x,y)=$

0. Let $\sigma : \widetilde{V} \longrightarrow V$ be the minimal resolution of (V, P). We fix a natural compactification $\bar{\sigma} : \widehat{V} \longrightarrow \bar{V}$ of σ. There exists an effective divisor Z_P supported on $\bar{\sigma}^{-1}(P)$ such that $\omega_{\widehat{V}} = \bar{\sigma}^*\omega_{\bar{V}} \otimes \mathcal{O}_{\widehat{V}}(-Z_P)$. Then we have

$$\chi(\mathcal{O}_{\widehat{V}}) - \chi(\mathcal{O}_{\bar{V}}) = p_g(P), \quad \omega_{\widehat{V}}^2 - \omega_{\bar{V}}^2 = -Z_P^2,$$

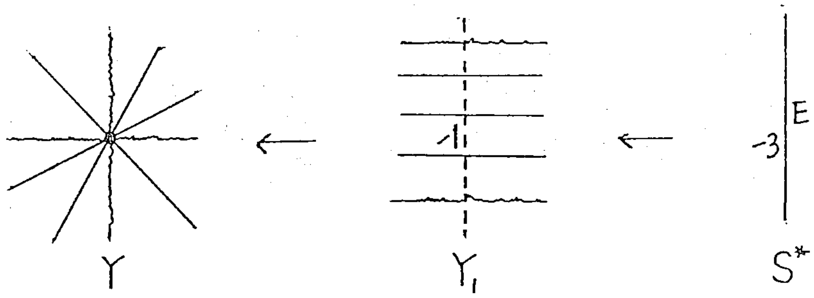
where $p_g(P)$ is the geometric genus of (V, P). We call $(p_g(P) : -Z_P^2)$ the type of singularity (V, P). For constructing a resolution process and for calculatiing the type of singularity (V, P), the method in §1 is useful. (Compare [15] and [18, §2].)

In this section, we give three examples. In the figures below, a broken line (resp. a usual line) means a component of the first (resp. the second) assistant curve at each step from S to S_r. For instance, the curve E_3 on Y_4 in Example 2.2 has $\mathbb{Z}$-weighting (1, 3). A dotted line is a curve of type (ϕ). A brack spot is the point for blow up in the next step. The number beside a line is the self-intersection number of the curve.

For all these examples, the type of the singularity is $(1 : 3)$ by Proposition 1.6 .(See also [8].)

Example 2.1 $\xi^3 + xy\xi + x^3 + y^3$.

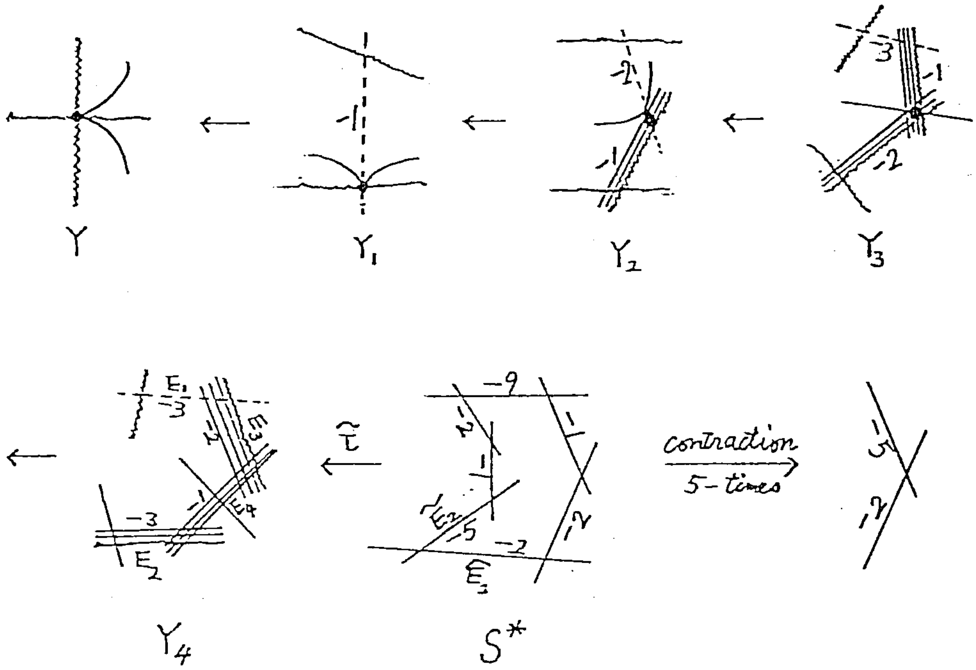
This is a simple elliptic singularity of type $\widetilde{E}_6$.([16]) Since the degree of Δ_1 is 6, E is an elliptic curve.



Example 2.2 $\xi^3 + xy\xi + x^5 + y^3$.

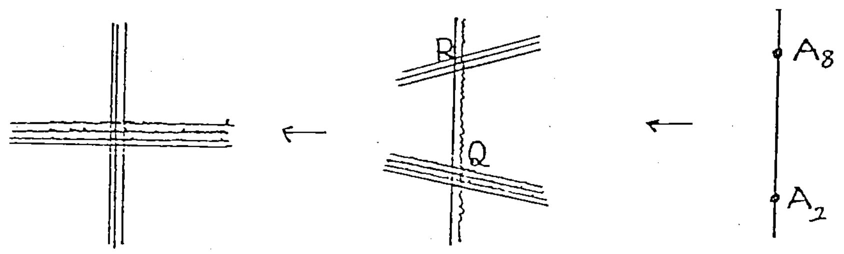
All the components of exceptional curves are rational. Over the curves E_2 and E_4 on Y_4, S_4 has compound nodes. The calculation for the self-intersection number is, for example, the following ; Since

$-3 = E_2^2 = (\tilde{\tau}_* \widetilde{E_2})\widetilde{E_2} = \widetilde{E_2}(\tilde{\tau}^* E_2) = \widetilde{E_2}(\widetilde{E_2} + 2\widehat{E_2}) = \widetilde{E_2}^2 + 2$, we have $\widetilde{E_2}^2 = -5$.



Example 2.3 $\quad \xi^3 + x^3\xi + xy^3$.

Since the equation over R is $\xi^3 + z\xi + y^3 z = z(\xi + y^3) + \xi^3 = z\zeta + (\zeta - y^3)^3 = \zeta(z + \cdots) - y^9$, this is a RDP of type A_8.



3 An inequality for a singularity $\xi^3 + h(x,y)$.

Let (V,P) be an isolated triple point. Assume that the equation of (V,P) is given by $\xi^3 + h(x,y) = 0$ after a suitable change of the coordinate. Let $\mu(P)$ be the Milnor number of (V,P).([11]) The aim in this section is to calculate the number $\mu(P) - 6p_g(P)$. (For the motivation, see Durfee [3, p.97].)

We prepare the following definition : In general, let Q be a point on a nonsingular surface. Let $F_1, \cdots, F_s$ be irreducible reduced curves passing through Q. For $s_1 \leq s$, we consider a divisor

$$D = \sum_{i=1}^{s_1} F_i + 2 \sum_{j=s_1+1}^{s} F_j.$$

Let t be the number of tangent line at Q of the reduced part D_{red} of D. We have automatically $t \leq s$. We set $m = mult_P(D)$ and $l = [m/3]$.

Definition 3.1 *We define the Durfee's number $d_Q(D)$ of D at Q by the following ;*
1) If $m \equiv 0$ (mod3) , then put $d_Q(D) = 3l(l+1) - 3s + 2s_1 - 2t + 3$.
2) If $m \equiv 1$ or 2 (mod 3), then put $d_Q(D) = 3l(l+1) - 3s + 2s_1 - 1$.

Now setting $G = \phi$ (empty) and $H = (h(x,y))$, we consider the diagram (2) in §1 of cyclic type at P. (See [2, §3].) The second assistant curve H_i coincides with the branch curve of $\pi_i : S_i \longrightarrow Y_i, 1 \leq i \leq r$. Let b be the number of irreducible components of H. Set $P_0 = P$ and $H_0 = H$.

Proposition 3.2 $\mu(P) - 6p_g(P) = \Sigma_{i=0}^{r-1} d_{P_i}(H_i) + b - 1$.

For the proof of this proposition, we use Laufer's formula [9].
For the Durfee's number $d_{P_i}(H_i)$, we have the following :

Lemma 3.3 *1) If $m_i = mult_{P_i}(H_i) \geq 6$ or $m_i = 3$, then we have $d_{P_i}(H_i) > 0$.*
2) If $m_i = 4$ or 3, then we have $d_{P_i}(H_i) \geq -2$. Moreover, if P_i is the point with $d_{P_i}(H_i) = -1$ (resp. $= -2$), then we have $\Sigma_{j=i+1}^{r-1} d_{P_j}(H_j) \geq 1$ (resp. ≥ 2).

Combining this with Tomari's result [19], we obtain the following :

Corollary 3.4 *Let (V, P) be an isolated singularity defined by $\xi^3 + h(x, y) = 0$. Then we have $\mu(P) \geq 6p_g(P)$.*

Problems 1) Show the analogous argument as in §3 for any 2-dimensional hypersurface triple point.

2) Construct the process of canonical resolution for surfaces of n-section type.

The detail will be published in our forthcoming paper.

References

[1] S. Abhyanker: Local rings of high embedded dimension, Amer. J. Math. 89 (1967), 1073-1077.

[2] T. Ashikaga and K. Konno: Examples of degenerations of Castelnuovo surfaces, preprint.

[3] A. Durfee: The signature of smoothings of complex surface singularities, Math. Ann. 232 (1978), 85-98.

[4] A. Franchetta: Sui punti doppi isolati delle superficie algebriche, Note I and II, Rend. Accad. dei Lincei (1946), 49-57, 162-168.

[5] E. Horikawa: On deformations of quintic surfaces, Invent. Math. 31 (1975), 43-85.

[6] E. Horikawa: Algebraic surfaces of general type with small c_1^2, I, II, III, IV, V, Ann. Math. 104 (1976), 358-387, Invent. Math. 37 (1976), 121-155, ibid. 47 (1978), 209-248, ibid. 50 (1979), 103-128, J. Fac. Sci. Univ. Tokyo 28 (1981), 745-755.

[7] D. Kirby: The structure of an isolated multiple point of a surface I, II, III, Proc. London Math. Soc. (3) VI (1956), 597-609, VII (1957), 1-28.

[8] H. Laufer: On minimally elliptic singularities, Amer. J. Math. 99 (1977), 1257-1295.

[9] H. Laufer: On μ for surface singularities, in *Proc. of Symposia in Pure Math.* 30 (1977), 45-49.

[10] H. Laufer: On normal two-dimensional double point singularities, Israel J. Math. 31 (1978), 315-334.

[11] J. Milnor: *Singular points of complex hypersurfaces*, Ann. of Math. Studies 61 (1968), Princeton.

[12] R. Miranda: Triple covers in algebraic geometry, Amer. J. Math. 107 (1985), 1123-1158.

[13] U. Persson: Double coverings and surfaces of general type, in *Algebraic geometry*, Lect. Notes in Math. 687 (1978), Springer.

[14] U. Persson: On Chern invariants of surfaces of general type, Compo. Math. 43 (1981), 3-58.

[15] M. Reid: Elliptic Gorenstein singularities of surfaces, preprint.

[16] K. Saito: Einfach-elliptische singularitäten, Invent. Math. 23 (1974), 289-325.

[17] M. Tomari: A geometric characterization of normal two-dimensional singularities of multiplicity two with $p_a \leq 1$, Publ. RIMS, Kyoto Univ. 20 (1984), 1-20.

[18] M. Tomari: A p_g-formula and elliptic singularities, Publ. RIMS, Kyoto Univ. 21 (1985), 297-354.

[19] M. Tomari: The inequality $8p_g < \mu$ for hypersurface two-dimensional isolated double points, preprint.

[20] Y. Xu and S.S.T. Yau: The inequality $\mu \geq 12p_g - 4$ for hypersurface weakly elliptic singularities, Comtemporary Math. 90 (1989), 317-344.

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A Remark on the Geography of Surfaces with Birational Canonical Morphisms

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Introduction. Let S be a minimal algebraic surface of general type defined over the field of complex numbers $\mathbf{C}$. Then by the inequalities of Noether and Bogomolov-Miyaoka-Yau, we have

$$2\chi(\mathcal{O}_S) - 6 \leq c_1^2(S) \leq 9\chi(\mathcal{O}_S),\ \chi(\mathcal{O}_S) > 0,\ c_1^2(S) > 0.$$

Thus a natural problem is to construct a surface S such that $\chi(\mathcal{O}_S) = x$ and $c_1^2(S) = y$ for any integers (x, y) in the above region. For this problem, Persson [15] proved that all the invariants (χ, c_1^2) with $2\chi - 6 \leq c_1^2 \leq 8\chi - 20$ occur for some surface carrying a pencil of curves of genus 2. Recently Xiao extended Persson's method and constructed many surfaces carrying pencils of hyperelliptic curves ([22],[24],[7]). We remark that the canonical maps of these surfaces are not birational because the relative canonical maps are already generically 2 : 1.

In this paper, we are interested in surfaces whose canonical maps are birational onto their images, because we believe that the "general" surface of general type has this property in view of the theory of curves. (For this motivation, see also [6], [23]). Since the lower bound of such surfaces is $c_1^2 = 3\chi - 10$ by Castelnuovo's second inequality ([3]), our problem is ;

"Construct surfaces with birational canonical map in the region $3\chi - 10 \leq c_1^2 \leq 9\chi$ naturally from the geographical point of view."

Many works are related to this problem. For instance, see [10],[5],[14], [25],[19], [8] etc.(Consult [16] for other references.)

Now our approach is in some sence a nonhyperelliptic version of Persson's original one. His way was natural in the following sence: For any surface S on the Noether line $c_1^2 = 2\chi - 6(\chi \geq 7)$, Horikawa [12, I] proved that S has a genus 2 fibration such that the "degenerate" genus 2 fibers come from the resolution of rational double points in some sence. So it seems that the simplest way to have many surfaces beyond the Noether line is to construct on S degenerate genus 2 fibers which come from the simple Gorenstein singularities not equal to RDP. Persson used the simple elliptic singularities of type $\widetilde{E_8}$.

Thus for our purpose, we first observe the surface on the Castelnuovo line $c_1^2 = 3\chi - 10$. By the classical idea of Castelnuovo [4], such a surface with birational canonical map($\chi \geq 9$) always has a fibration of nonhyperelliptic curves of genus 3 and the "degenerate" genus 3 fibers come from the resolution of RDP. Therefore it is natural for our purpose to construct many simple degenerate genus 3 fibers which do not come from RDP. We use simple elliptic singularities of type $\widetilde{E_7}$. Our result is the following :

Theorem *Let x, y be any integers with $3x - 10 \leq y \leq 8x - 78$. Then there exists a minimal surface S such that (1) $\chi(\mathcal{O}_S) = x, c_1^2(S) = y$, (2) the canonical map of S is a birational holomorphic map onto its image and (3) S has a fibration of nonhyperelliptic curves of genus 3.*

One can also find such surfaces in a subregion of $8\chi - 77 \leq c_1^2 \leq 8\chi - 35$(see §3). The key step of the proof is to study the resolution process of singularities arising from certain cyclic quadruple coverings.

Acknowledgement and Notes: I express my special thanks to Dr. Kazuhiro Konno. Konno and I held seminars every week in 1988/89, and he gave me many essential advices. Now, after his recent beautiful work [13] was established (December 1989), the approach here is not "legitimate" in some sence (see [13, Remark 9.9]). For more natural way to this problem, please see his appendix to this paper.

I also thank Prof. Eiji Horikawa who communicated with me on the occasion of his visit in Sendai (March 1989). He had already studied surfaces carrying pencils of curves of genus 3 in detail about ten years ago. But his paper have not appeared yet unfortunately. He also gave us the question about the region of existence of such surfaces. Considering our result, one can ask;

"If S has a pencil of nonhyperelliptic curves of genus 3 , then do we have $3\chi(\mathcal{O}_S) - 10 \leq c_1^2(S) \leq 8\chi(\mathcal{O}_S)$?? "

I also thank Prof. Sampei Usui who explained me the relation between our method and a way to the Torelli problem and inspired me. I also thank Dr. Noboru Nakayama who gave me useful comment on irregularity.

1 Construction.

1.1 Let m, n be nonnegative integers satisfying

$$m + n \geq 2, \ \ n \leq 3. \tag{1}$$

Let C be a nonsingular curve of genus g, and let D_m, D_n be divisors of degree m, n on C respectively. We let

$$\delta : W = \mathbf{P}(\mathcal{O}_C \bigoplus \mathcal{O}_C(D_m)) \longrightarrow C$$

be the geometrically ruled surface, and denote by C_0, C_∞ and f the 0-section ($C_0^2 = m$), the ∞-section and a fiber on W respectively. We set

$$\pi : X = \mathbf{P}(\mathcal{O}_W \bigoplus \mathcal{O}_W(C_0)) \longrightarrow W$$

the $\mathbf{P}^1$-bundle, and set $T = \mathcal{O}_X(1)$.

Let S be an irreducible divisor on X which is linearly equivalent to $4T + \pi^*\delta^* D_n$. We denote by ω_S the dualizing sheaf on S, and set $p_g(S) = h^0(S, \omega_S)$ and $q(S) = h^1(S, \mathcal{O}_S)$.

Lemma 1.2
$p_g(S) = 4m + 3n + 3(g-1), \ q(S) = 0, \ \omega_S^2 = 12m + 8n + 16(g-1).$

Proof. Denoting by K_X the canonical bundle on X, we have $K_X + S \sim -2T + \pi^*(K_W + S) + 4T + \pi^*\delta^* D_n \sim 2T + \pi^*(-C_0 + \delta^*(K_C + D_m + D_n))$ where "$\sim$" is the linear equivalence. Thus we have $h^0(X, K_X + S) = \Sigma_{i=0}^2 h^0(W, (i-1)C_0 + \delta^*(K_C + D_m + D_n)) = 2h^0(C, K_C + D_m + D_n) + h^0(C, K_C + 2D_m + D_n) = 4m + 3n + 3(g-1)$ by (1). Since

$R^p\pi_*\mathcal{O}(K_X+S)=0$ for $p\geq 1$, we have similarly $H^p(X,K_X+S)=0$ for $p=0,1$. Hence by the long exact sequence associated with

$$0\longrightarrow \mathcal{O}(K_X)\longrightarrow \mathcal{O}(K_X+S)\longrightarrow \mathcal{O}(\omega_S)\longrightarrow 0,$$

we have $p_g(S)=h^0(X,K_X+S)$ and $q(S)=h^2(X,K_X)=h^1(C,\mathcal{O}_C)=g$.

Since $\omega_S^2=(K_X+S)^2S$ and $T^2=c_1(\mathcal{O}_W\oplus\mathcal{O}_W(C_0))T$, we also have the desired formula for ω_S^2 by an easy calculation. *q.e.d.*

1.3 From now on, we fix a fiber coordinate (Y_0,Y_1) of $\pi:X\longrightarrow W$. Denote by T_∞ the section defined by $Y_1=0$. T_∞ is linearly equivalent to $T-\pi^*C_0$. If we choose D_n general, then the divisor $S|_{T_\infty}$ (the restriction of S to T_∞) is n distinct fibers on $T_\infty\simeq W\longrightarrow C$. On the other hand, the self-intersection number of the divisor $T_\infty|_S$ on S is $-n$. Thus S contains n exceptional curves of first kind. Let $\nu:S\longrightarrow S'$ be the contraction of these curves. We remark that ν is induced by the contraction of the ambient space

$$\Phi_T:X\longrightarrow X':=\mathbb{P}(\mathcal{O}_C\bigoplus\mathcal{O}_C\bigoplus\mathcal{O}_C(D_m)),$$

where Φ_T is the morphism defined by the linear system $|T|$. Then we have $p_g(S')=p_g(S),q(S')=q(S)$ and $\omega_{S'}^2=\omega_S^2+n=3\chi(\mathcal{O}_{S'})+10(g-1)$ by Lemma 1.2.

Now assume that (1) S has k simple elliptic singularities of type $\widetilde{E_7}$ (see [18]), (2) other singularities of S are at most rational double points (RDP for short) and (3) S is smooth along $T_\infty|_S$. Let $\tilde{S}$ be the minimal resolution of singularities of S'. Then by [17] or [1] we have

$$\chi(\mathcal{O}_{\tilde{S}})=4m+3n+2(g-1)-k,\ \omega_{\tilde{S}}^2=12m+9n+16(g-1)-2k.$$

In order to realize the above assumption, we set up the following situation.

Considering that any $\phi\in H^0(X,4T+\pi^*\delta^*D_n)$ can be written as $\phi=\sum_{i=0}^4\phi_{iC_0+\delta^*D_n}Y_0^{4-i}Y_1^i$ where $\phi_{iC_0+\delta^*D_n}\in H^0(W,iC_0+\delta^*D_n)$, we set

$$|4T+\pi^*\delta^*D_n|_C=\left\{(\phi)\in|4T+\pi^*\delta^*D_n|;\phi_{\delta^*D_n}Y_0^4+\phi_{4C_0+\delta^*D_n}Y_1^4\right\},$$

where (ϕ) is the divisor defined by ϕ. We call it the *cyclic subsystem* of $|4T + \pi^*\delta^*D_n|$. For any member $(\phi) \in |4T + \pi^*\delta^*D_n|_C$, we call the divisor $B_\phi = (\phi_{4C_0+\delta^*D_n})$ and $B'_\phi = (\phi_{\delta^*D_n})$ the branch and the assistant bracch locus of (ϕ), respectively. With this notation, we define the *good cyclic subsystem* $|4T + \pi^*\delta^*D_n|_{GC}$ by the following two properties:

1) B_ϕ is reduced and B'_ϕ is nonsingular.
2) B'_ϕ passes through no singular points of B_ϕ, and they meet transversally.

Lemma 1.4 *Let $S = (\phi)$ be a member of $|4T + \pi^*\delta^*D_n|_{GC}$ and set $\pi_S := \pi|_S : S \longrightarrow W$. Then we have*

(a) S is irreducible and normal,

(b) $Sing(S) = \pi_S^{-1}(Sing(B_\phi)) \cap Support(T)$,

(c) the general fiber of π_S is a nonhyperelliptic curve of genus 3, where $Sing(S)$ is the singular locus of S, etc.

Proof On $\pi_S^{-1}(W \backslash B_\phi \cap B'_\phi)$, π_S is a finite morphism and the assertion (b) is clear. For any $P \in B_\phi \cap B'_\phi$, denote by (x, y) a local (analytic) parameter at P on W. Then π_S^{-1} is defined by the equation $xY_0^4 + yY_1^4 = 0$ in X. Thus S is nonsingular along $\pi_S^{-1}(B_\phi \cap B'_\phi)$. ($\pi_S^{-1}(P)$ is a nonsingular (-4) curve.) Hence (b) holds. Especially S is normal.

Let $\tilde{\tau} : \tilde{S} \longrightarrow S$ be the resolution of S and $\tilde{S} = \cup_i S^{(i)}$ be the decomposition to its connected components. Since $B_\phi \cap B'_\phi$ is a set of points, $\pi_S\tilde{\tau}(S^{(i)})$ coincides with W for any i. On the other hand, for a general point Q of B_ϕ, $\pi_S^{-1}(Q)$ consists of one point and nonsingular in S. Hence $i = 1$, i.e. S is irreducible.

The assertion (c) is clear. *q.e.d.*

Definition 1.5 A curve singularity P is called an *infinitely close double point* iff it consists of two tangent branches no longer simultaneously tangent after one blow up. (See the point P_i in Fig. 1).

1.6 Let S be a surface defined by $(\phi) \in |4T + \pi^*\delta^*D_n|_{GC}$. If $P \in B_\phi \backslash B'_\phi$ is an infinitely close double point (resp. an ordinaly double point) of B_ϕ, then $Q = \pi_S^{-1}(P)$ is an $\widetilde{E_7}$-singularity (resp. a RDP of type A_3). Therefore we try to construct a member $B := B_\phi \in |4C_0 + \delta^*D_n|$ such that B has k infinitely close double points and other singularities of B are at most ordinary double points. The following method is essentially due to Persson [15].

Let D_β be a divisor of degree $\beta \geq 0$ on C. Assume that there is a divisor $D_r \geq 0$ (effective or zero) on C such that

(2) $$D_r \sim D_n - 2D_\beta.$$

Let $P_1, \cdots, P_k$ be points on C_0 such that

(3) $$h^0(C, 2D_m + D_\beta - \widehat{P_1} - \cdots - \widehat{P_k}) \geq 2,$$

where $\widehat{P_i} = \delta(P_i)$. Let $B^{(1)}$ and $B^{(2)}$ be general members of Persson's subsystem $|2C_0 + \delta^*D_\beta|_P$ ([1]) passing through $P_1, \cdots, P_k$. Assume that

(4) *δ^*D_r does not pass through the singular points of the reducible curve $B^{(1)} + B^{(2)}$.*

We set

$$B := B^{(1)} + B^{(2)} + \delta^*D_r.$$

Then B is linearly equivalent to $4C_0 + \delta^*D_n$, B has infinitrely close double points at $P_i (1 \leq i \leq k)$ and other singularities of B are at most ordinaly singular points (see the argument in [1]). From the above argument :

Proposition 1.7 *Let m, n be as in (1). Assume that*
(a) $0 \leq k \leq 2m + [n/2] - 1$ when $g = 0$,
(b) $0 \leq k \leq 2m + [n/2] - 2$ when $g \geq 1$,
where $[n/2]$ is the greatest integer not exceeding $n/2$. Then there exists a nonsingular surface $\tilde{S}$ such that
1) $\chi(\mathcal{O}_{\tilde{S}}) = 4m + 3n + 2(g-1) - k$, $c_1^2(\tilde{S}) = 12m + 9n + 16(g-1) - 2k$,
2) $\tilde{S}$ has a (base point free) pencil of nonhyperelliptic curves of genus 3 over a curve of genus g.

Proof When $C = \mathbf{P}^1$: By (a), we can choose D_β and D_γ such that the conditions (2), (3) and (4) are satisfied.

When $g(C) \geq 1$: Let C be a hyperelliptic curve of genus $g \geq 1$ and let $\iota : C \longrightarrow \mathbf{P}^1$ be the hyperelliptic involution. We choose divisors D_m such that ;

(c) When m is even, $D_m := \iota^* \mathcal{O}_{\mathbf{P}^1}(m/2)$,

(d) When m is odd, $D_m := \iota^* \mathcal{O}_{\mathbf{P}^1}([m/2]) + Q$ for a ramification point $Q \in C$.

We choose divisors D_n, D_β, D_γ and $\widehat{P_1} + \cdots + \widehat{P_k}$ in a similar way carefully. Then the conditions (2), (3) and (4) are satisfied by (b).

q.e.d.

Remark 1.8 In the proof of Proposition 1.7, if C is a general curve of genus $g \geq 1$ and D_m, D_n are general divisors on C, then the invariants (χ, c_1^2) of the resulting surfaces $\tilde{S}$ do not cover wider area than the above ones in the surface geography.

2 Process of resolution.

In this section, first we resolve singularities of S explicitely. This method is analogous to that in [11, §2] and [2, §3]. Second we calculate the invariants of surfaces in each process of the resolution.

2.1 We go back to the situation in *1.6*. Our way of resolving singularities of S consists of the following six steps :

1) Let $f_1, \cdots, f_k$ be the fibers of $W \longrightarrow C$ containing $P_1, \cdots, P_k$ respectively. Let $\tau_1 : W_1 \longrightarrow W$ be the composition of blow ups at $P_i (1 \leq i \leq k)$ and set $g_i' = \tau_1^{-1}(P_i)$. Denote by f_i' the strict transform of f_i and by Q_i' the point on intersection of f_i' and g_i'.

2) Let $\tau_2 : W_2 \longrightarrow W_1$ be the composition of blow ups at $Q_i', 1 \leq i \leq k$. Set $h_i'' = \tau_1^{-1}(Q_i')$ and denote by f_i'' and g_i'' the strict transform of f_i'

and g'_i by τ respectively. Put $\tau = \tau_2 \circ \tau_1$. We set

$$L_2 = \tau^* \mathcal{O}_W(C_0) \bigotimes \mathcal{O}_{W_2}(-\sum_{n=1}^{k} h''_i),$$
$$B_2 = \tau^* B_\phi - 4\sum_{i=1}^{k} h''_i = \widetilde{B_2} + 2\sum_{i=1}^{k} g''_i, \quad B'_2 = \tau^* B'_\phi,$$

where $\widetilde{B_2}$ is the strict transform of B_ϕ by τ. Put

$$\pi_2 : X_2 = \mathbf{P}(\mathcal{O}_{W_2} \bigoplus \mathcal{O}_{W_2}(L_2)) \longrightarrow W_2$$

and set $T_2 = \mathcal{O}_{X_2}(1)$. We define a member S_2 of the cyclic subsystem $|4T_2 + \pi_2^* \tau^* \delta^* D_n|_C$ such that the branch and the assistant branch locus are B_2 and B'_2 respectively. Then by the same argument as in [2, §3], there is a natural morphism $\mu : X_2 \longrightarrow X$ such that $\widehat{\mu} = \mu|_{S_2} : S_2 \longrightarrow S$ is a birational morphism. The singular locus of S_2 (except RDP) coincides with $\cup_{i=1}^{k} (\pi_2|_{S_2})^{-1}(g''_i)$. The local analytic equation of this is of the form $\xi^4 + x^2 = 0$. We set $\widetilde{g_i} = (\pi_2|_{S_2})^{-1}(g''_i)_{red}$ the reduced curve of $(\pi_2|_{S_2})^{-1}(g''_i)$.

3) Let $\mu_3 : X_3 \longrightarrow X_2$ be the composition of blow ups at $\widetilde{g_i}, 1 \leq i \leq k$. Denote by S_3 the strict transform of S_2 by μ_3, and set $E_i = \mu_3^{-1}(\widetilde{g_i})$, $\widehat{g_i} = (S_3 \cap E_i)_{red}$ and $\widehat{\mu_3} = \mu|_{S_3}$.

4) Let $\mu_4 : X_4 \longrightarrow X_3$ be the composition of blow ups at $\widehat{g_i}, 1 \leq i \leq k$. Denote by S_4 and E'_i the strict transforms of S_3 and E_i by τ_4 respectively. Set $\widetilde{E_i} = \mu_4^{-1}(\widehat{g_i})$ and $\widehat{\mu_4} = \mu_4|_{S_4}$. S_4 contains $2k$ exceptional curves of first kind $(\widehat{\mu_4})^{-1}(\widehat{g_i}), 1 \leq i \leq k$.

5) Let $\mu_5 : S_4 \longrightarrow S_5$ be the composition of contraction maps of $(\widehat{\mu_4})^{-1}(\widehat{g_i})$ for $1 \leq i \leq k$ and of n exceptional curves $T_\infty|_S$. Let h'''_i be the image by μ_5 of the strict transform of h''_i by $\widehat{\mu_4} \circ \widehat{\mu_3}$. h'''_i is a nonsingular elliptic curve of self-intersection number (-2). (see Fig. 1)

6) Let $\mu_6 : \tilde{S} \longrightarrow S_5$ be the minimal resolution of all RDP on S_5. $\tilde{S}$ is nonsingular.

By the above process, we obtain the following diagram :

$$
\begin{array}{ccccccccccc}
X & \xleftarrow{\mu} & X_2 & \xleftarrow{\mu_3} & X_3 & \xleftarrow{\mu_4} & X_4 & & & & \\
\cup & & \cup & & \cup & & \cup & & & & \\
S & \xleftarrow{\widehat{\mu}} & S_2 & \xleftarrow{\widehat{\mu_3}} & S_3 & \xleftarrow{\widehat{\mu_4}} & S_4 & \xrightarrow{\mu_5} & S_5 & \xleftarrow{\mu_6} & \widehat{S} \\
\downarrow \pi_S & & \downarrow \pi_{S_2} & & & & & & & & \\
W & \xleftarrow{\tau} & W_2 & & & & & & & &
\end{array}
$$

(Fig. 1)

(Fig. 2)

Next for calculating $p_g(S_j)$ and $q(S_j)$ for $2 \le j \le 4$, we prepare the following two lemmas.

Lemma 2.2 *We set*
$\Theta := k - h^0(C, K_C + 2D_m + D_n) + h^0(C, K_C + 2D_m + D_n - \widehat{P_1} \cdots - \widehat{P_k})$.
Then we have

(a) $h^p(W_2, \tau^*(-C_0 + \delta^*(K_C + D_m + D_n)) + \Sigma_{i=1}^k (g_i'' + h_i'')) = 0, (\forall p)$,
(b) $h^p(W_2, \tau^*\delta^*(K_C + D_m + D_n) + \Sigma_{i=1}^k g_i'')$

$$
= \begin{cases} m + n + g - 1 & (p = 0) \\ k & (p = 1) \\ 0 & (p = 2) \end{cases}
$$

(c) $h^p(W_2, \tau^*(C_0 + \delta^*(K_C + D_m + D_n)) + \Sigma_{i=1}^k (g_i'' - h_i''))$

$$
= \begin{cases} 3m + 2n + 2(g-1) - k + \Theta & (p = 0) \\ 2k + \Theta & (p = 1) \\ 0 & (p = 2) \end{cases}
$$

Proof. We write $g_i = g_i'', h_i = h_i''$ and $\Sigma = \Sigma_{i=1}^k$ for simplicity.
(a) : Put $D_1 = \tau^*(-C_0 + \delta^*(K_C + D_m + D_n))$, and consider the exact sequences

$$
\begin{aligned}
0 \longrightarrow \mathcal{O}_{W_2}(D_1) &\longrightarrow \mathcal{O}_{W_2}(D_1 + \textstyle\sum(g_i + h_i)) \\
&\longrightarrow \mathcal{O}_{\sum(g_i + h_i)}(D_1 + \textstyle\sum(g_i + h_i)) \longrightarrow 0,
\end{aligned}
$$

$$
\begin{aligned}
0 &\longrightarrow \mathcal{O}_{\sum h_i}(D_1+\textstyle\sum h_i) \longrightarrow \mathcal{O}_{\sum(g_i+h_i)}(D_1+\sum(g_i+h_i)) \\
&\longrightarrow \mathcal{O}_{\sum g_i}(D_1+\textstyle\sum(g_i+h_i)) \longrightarrow 0,
\end{aligned}
$$

$$
\begin{aligned}
0 &\longrightarrow \mathcal{O}_{\sum_{i=1}^{j-1} h_i}(D_1+\sum_{i=1}^{j-1} h_i) \longrightarrow \mathcal{O}_{\sum_{i=1}^{j} h_i}(D_1+\sum_{i=1}^{j} h_i) \\
&\longrightarrow \mathcal{O}_{h_j}(D_1+\sum_{i=1}^{j} h_i) \longrightarrow 0,
\end{aligned}
$$

for $2 \leq j \leq k$. By using

$$
(D_1+\textstyle\sum h_i)h_j = (D_1+\sum(g_i+h_i))g_j = -1
$$

inductively, we have

$$
\begin{aligned}
&H^p(W_2, D_1+\Sigma(g_i+h_i)) \simeq H^p(W_2, D_1) \\
&\simeq H^p(W, -C_0+\delta^*(K_C+D_m+D_n)) = 0,
\end{aligned}
$$

for any p.

(b) : Since

$$
(\tau^*\delta^*(K_C+D_m+D_n)+\textstyle\sum g_i)g_j = -2,
$$

we obtain (b) similarly.

(c) : Set $D_2 = \tau^*(C_0+\delta^*(K_C+D_m+D_n))$ and consider the exact sequences

$$
\begin{aligned}
0 &\longrightarrow \mathcal{O}_{W_2}(D_2-\textstyle\sum h_i) \longrightarrow \mathcal{O}_{W_2}(D_2+\sum(g_i-h_i)) \\
&\longrightarrow \mathcal{O}_{\sum g_i}(D_2+\textstyle\sum(g_i-h_i)) \longrightarrow 0,
\end{aligned}
$$

$$
0 \longrightarrow \mathcal{O}_{W_2}(D_2-\textstyle\sum h_i) \longrightarrow \mathcal{O}_{W_2}(D_2) \longrightarrow \mathcal{O}_{\sum h_i} \longrightarrow 0.
$$

Since $(D_2+\Sigma(g_i-h_i))g_j = -3$ and $D_2|_{h_j} \simeq \mathcal{O}_{h_j}$, we have

$$
H^0(W_2, D_2-\textstyle\sum h_i) \simeq H^0(W_2, D_2+\sum(g_i-h_i)),
$$

$$
\begin{aligned}
(5) \qquad 0 &\longrightarrow H^1(W_2, D_2-\Sigma h_i) \longrightarrow H^1(W_2, D_2+\Sigma(g_i-h_i)) \\
&\longrightarrow \mathbf{C}^{2k} \longrightarrow 0 \ (exact),
\end{aligned}
$$

$$\begin{aligned} 0 &\longrightarrow H^0(W_1, D_2 - \sum h_i) \longrightarrow H^0(W_1, D_2) \xrightarrow{\mathbf{r}} \mathbf{C}^k \\ &\longrightarrow H^1(W^1, D_2 - \sum h_i) \longrightarrow H^1(W_1, D_2) \longrightarrow 0 \ (exact). \end{aligned}$$

We denote by Θ' the dimension of the cokernel of the map $\mathbf{r}$. Then by (5), we have

$$(6) \quad \begin{aligned} &h^0(W_2, D_2 + \Sigma(g_i - h_i)) \\ &\quad = h^0(W, C_0 + \delta^*(K_C + D_m + D_n)) - k + \Theta', \\ &h^1(W_2, D_2 + \Sigma(g_i - h_i)) \\ &\quad = h^1(W, C_0 + \delta^*(K_C + D_m + D_n)) + 2k + \Theta'. \end{aligned}$$

Now we consider the map $\mathbf{r}$.

Any element $\psi' \in H^0(W_2, D_2)$ is written as $\psi' = \tau^*\psi$ for some $\psi \in H^0(W, C_0 + \delta^*(K_C + D_m + D_n))$. Then we have

$$\mathbf{r}(\psi') = (\psi'|_{h_1}, \cdots, \psi'|_{h_k}) = (\psi(P_1), \cdots, \psi(P_k)) \in \mathbf{C}^k$$

where $\psi'|h_i \in H^0(h_i, D_2|_{h_i}) \simeq \mathbf{C}$ and $\psi(P_i)$ is the value of ψ at P_i. We set $M = H^0(W, C_0 + \delta^*(K_C + D_m + D_n))$ and denote by M^i the vector subspace of M consisting of the elements which pass through $P_1, \cdots, P_i$. We have a descending filtration

$$(7) \qquad M = M^0 \supset M^1 \supset \cdots \supset M^k.$$

We have dim M^i = dim M^{i-1} or dim M^i = dim $(M^{i-1}) - 1$ according as P_i is a base point of M^{i-1} or not. Then the number Θ' is equal to the cardinality of i such that dim M^i = dim M^{i-1}.

Let $(Z_0 : Z_1)$ be a fiber coordinate of $\pi : W \longrightarrow C$ such that $C_0 = \{Z_0 = 0\}$. Then any $\psi \in H^0(W, C_0 + \delta^*(K_C + D_m + D_n))$ is written by

$$\psi = \psi_{K_C+D_m+D_n} Z_0 + \psi_{K_C+2D_m+D_n} Z_1$$

where $\psi_{K_C+jD_m+D_n} \in H^0(C, K_C + jD_m + D_n)$ for $j = 1, 2$. Since $P_i (1 \leq i \leq k)$ is on C_0, $\psi(P_i) = 0$ is equivalent to $\psi_{K_C+2D_m+D_n}(\widehat{P_i}) = 0$. We set $N = H^0(C, K_C + 2D_m + D_n)$ and denote by N^i the subspace of N consisting of the elements which pass through $\widehat{P_1}, \cdots, \widehat{P_i}$. We have a filtration

$$(8) \qquad N = N^0 \supset N^1 \supset \cdots \supset N^k.$$

By the above argument, the number Θ' is equal to the cardinality of i such that $\dim N^i = \dim N^{i-1}$. Hence Θ' coincides with Θ. Thus by (6), we obtain (c). *q.e.d.*

Lemma 2.3 *For any j $(1 \leq j \leq k)$, we have*
(a) $(K_{X_2} + S_2)\widetilde{g_j} = -3$,
(b) $(K_{X_3} + S_3)\widehat{g_j} = -2$.

Proof. Denote by T_2^∞ the ∞-section of $\pi_2 : X_2 \longrightarrow W_2$. Then we have

$$(9) \qquad T_2\widetilde{g_j} = (T_2^\infty + \pi_2^*(\tau^*C_0 - \sum_{i-1}^{k} h_i'')\widetilde{g_j} = -h_j''g_j'' = -1.$$

Since

$$(10) \qquad \begin{aligned} K_{X_2} + S_2 &\sim -2T_2 + \pi_2^*(K_{W_2} + L_2) + 4T_2 + \pi_2^*\tau^*\delta^*D_n \\ &\sim 2T_2 + \pi_2^*(\tau^*(-C_0 + \delta^*(K_C + D_m + D_n) + \Sigma(g_i'' + h_i'')), \end{aligned}$$

we obtain (a).

For $1 \leq i \leq k$, let $\widetilde{T_2}, \widetilde{G_i}, \widetilde{H_i}$ and $\widetilde{F_i}$ be the proper transforms of $T_2, G_i = \pi_2^{-1}(g_i''), H_i = (\pi_2'')^{-1}(h_i'')$ and $F_i = \pi_2^{-1}(f_i'')$ by μ_3 respectively (see Fig. 2). Then by (10), we have

$$(11) \qquad \begin{aligned} (K_{X_3} + S_3)\widehat{g_j} &= (\mu_3^*(K_{X_2} + S_2) - \Sigma_{i=1}^k E_i)\widehat{g_j} \\ &= (2(\widetilde{T_2} + \Sigma E_i) + \Sigma(\widetilde{G_i} + E_i + \widetilde{H_i}) - \Sigma E_i)\widehat{g_j} \\ &= 2E_j\widehat{g_j} + \widetilde{G_j}\widehat{g_j} + 1. \end{aligned}$$

Now we consider the exact sequence

$$0 \longrightarrow N_{\widetilde{g_i}/T_2} \longrightarrow N_{\widetilde{g_j}/X_2} \longrightarrow (N_{T_2/X_2})|_{\widetilde{g_j}} \longrightarrow 0,$$

where $N_{\widetilde{g_j}/T_2}$ is the normal bundle of $\widetilde{g_j}$ in T_2, etc. We have $N_{\widetilde{g_j}/X_2} \simeq \mathcal{O}_{\mathbf{P}^1}(-2)$ and $(N_{T_2/X_2})|_{\widetilde{g_j}} \simeq \mathcal{O}_{\mathbf{P}^1}(-1)$ by(9). Since $Ext^1(\mathcal{O}(-1), \mathcal{O}(-2))$ vanishes, we have $N_{\widetilde{g_j}/X_2} \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-2)$. Thus E_j is isomorphic to the Hirzebruch surface of degree 1, and $\widehat{g_j}$ is the ∞-section of E_j. Hence we have $\widetilde{G_j}\widehat{g_j} = (\widetilde{G_j}|_{E_j})^2 = -1$.

On the other hand, the divisor $E_j + \widetilde{G_j} + 2\widetilde{H_j} + \widetilde{F_j}$ is a singular fiber of the degeneration $X_3 \longrightarrow C$. Thus we have

$$0 = (E_j + \widetilde{G_j} + 2\widetilde{H_j} + \widetilde{F_j})\widehat{g_j} = E_j\widehat{g_j} + 1.$$

Hence by (11), we obtain (b). *q.e.d.*

Proposition 2.4 *For $2 \leq j \leq 4$, we have ;*

$$
\begin{aligned}
&1)\ H^0(\omega_{S_j}) \simeq H^0(X_j, K_{X_j}+S_j),\\
&\quad\ h^1(\omega_{S_j}) = h^1(X_j, K_{X_j}+S_j) - h^2(X_j, K_{x_j}+S_j) + g\ (\forall j),\\
&2)\ h^0(\omega_{S_2}) = h^0(\omega_S) - k + \Theta,\ \ h^1(\omega_{S_2}) = g + 3k + \Theta,\\
&3)\ H^0(\omega_{S_3}) \simeq H^0(\omega_{S_2}),\ \ h^1(\omega_{S_3}) = h^1(\omega_{S_2}) - 2k,\\
&4)\ H^0(\omega_{S_4}) \simeq H^0(\omega_{S_3}),\ \ h^1(\omega_{S_4}) = h^1(\omega_{S_3}) - k.
\end{aligned}
$$

Proof. Since $H^p(K_{X_j}) = 0 (p = 0,1), h^2(K_{X_j}) = g$ and $H^2(\omega_{S_j}) \simeq H^3(K_{X_j}) \simeq \mathbf{C}$, the long exact sequence associated with

$$0 \longrightarrow \mathcal{O}(K_{X_j}) \longrightarrow \mathcal{O}(K_{X_j}+S_j) \longrightarrow \mathcal{O}(\omega_{S_j}) \longrightarrow 0,$$

implies 1).

By (10), we have $R^p(\pi_2)_*\mathcal{O}(K_{X_2}+S_2) = 0$ for $p \geq 1$. Thus we have

$$
\begin{aligned}
&H^p(X_2, K_{X_2}+S_2) \simeq H^p(W_2, (\pi_2)_*\mathcal{O}(K_{X_2}+S_2))\\
&\simeq H^p(W_2, \tau^*(-C_0 + \delta^*(K_C + D_m + D_n)) + \Sigma_i(g_i + h_i))\\
&\ \oplus H^p(W_2, \tau^*\delta^*(K_C + D_m + D_n) + \Sigma_i g_i)\\
&\ \oplus H^p(W_2, \tau^*(C_0 + \delta^*(K_C + D_m + D_n)) + \Sigma_i(g_i - h_i)).
\end{aligned}
$$

Hence by Lemma 2.2 , we obtain 2).

Since $K_{X_3} \sim \mu_3^* K_{X_2} + \Sigma_i E_i$ and $S_3 \sim \mu_3^* S_2 - 2\Sigma_i E_i$, we have the exact sequence

$$
\begin{aligned}
(12)\qquad 0 &\longrightarrow \mathcal{O}(K_{X_3}+S_3) \longrightarrow \mathcal{O}(\mu_2^*(K_{X_2}+S_2))\\
&\longrightarrow \mathcal{O}_{\sum E_i}(\mu_3^*(K_{X_3}+S_3)|_{\sum E_i}) \longrightarrow 0.
\end{aligned}
$$

By Lemma 2.3 (a), we have

$$H^p(E_i, \mu_3^*(K_{X_2}+S_2)|_{E_i}) \simeq H^p(\mathbf{P}^1, \mathcal{O}(-3)) \simeq \begin{cases} 0 & (p \neq 1) \\ \mathbf{C}^2 & (p = 1) \end{cases}$$

Thus by the long exact sequence associated with (12) and by 1), we obtain 3).

By Lemma 2.3 (b), we obtain 4) similarly. *q.e.d.*

Corollary 2.5 $p_g(\tilde{S}) = p_g(S) - k, \quad q(\tilde{S}) = g.$

Proof. By Proposition 2.4, we have $p_g(\tilde{S}) = p_g(S_4) = p_g(S) - k + \Theta$ and $q(\tilde{S}) = q(S_4) = g + \Theta$. On the other hand, Θ equals to zero by Proposition 1.7. *q.e.d.*

3 Canonical mapping.

In this section, we study the canonical mapping of the surface $\tilde{S}$ constructed in the previous section, and prove our theorem.

Proposition 3.1 *Assume that*
(a) $m + n \geq 3$ *and* $2m + n + 2g - 2 \geq k$, *or*
(b) $C = \mathbf{P}^1, m + n = 2$ *and* $m - 1 \geq k$.
Then $\tilde{S}$ *is relatively minimal and the canonical map* $\Phi_{K_{\tilde{S}}}$ *of* $\tilde{S}$ *is a birational holomorphic map onto its image.*

Proof. Since $H^0(K_{\tilde{S}})$ is naturally isomorphic to $H^0(X_2, K_{X_2} + S_2)$ by Proposition 2.4, we consider this space. The divisor $K_{X_2} + S_2$ is linearly equivalent to $T_2^\infty + T_2 + \pi_2^*(\tau^*\delta^*(K_C + D_m + D_n) + \Sigma_{i=1}^k g_i)$, and T_2^∞ is a part of the fixed component of this system by the proof of Lemma 2.2(a). We set

$$D := T_2 + \pi_2^*(\tau^*\delta^*(K_C + D_m + D_n) + \sum_i g_i).$$

If we fix a fiber coordinate $(\widehat{Y_0} : \widehat{Y_1})$ of $\pi_2 : X_2 \longrightarrow W_2$, then any $\phi \in H^0(X_2, D)$ is written as

$$\phi = \phi^{(0)}\widehat{Y_0} + \phi^{(1)}\widehat{Y_1}, \tag{13}$$

where $\phi^{(0)} \in H^0(W_2, \tau^*\delta^*(K_C + D_m + D_n) + \Sigma g_i)$ and $\phi^{(1)} \in H^0(W_2, \tau^*(C_0 + \delta^*(K_C + D_m + D_n)) + \Sigma(g_i - h_i))$. By the proof of Lemma 2.2, we have $H^0(W_2, \tau^*\delta^*(K_C + D_m + D_n) + \Sigma g_i) \simeq H^0(W_2, \tau^*\delta^*(K_C + D_m + D_n)) \overset{\tau^*}{\simeq} H^0(W, \delta^*(K_C + D_m + D_n))$. Moreover $\phi^{(1)}$ is written as

$$\phi^{(1)} = \tau^*\psi, \quad \psi = \psi^{(0)}Z_0 + \psi^{(1)}Z_1, \tag{14}$$

where $\psi \in M^k \subset H^0(W, C_0 + \delta^*(K_C + D_m + D_n))$, $\psi^{(0)} \in H^0(C, K_C + D_m + D_n)$ and $\psi^{(1)} \in N^k$ (see (7) and (8)).

Hence by the expression (13) and (14), the curve $A = \Sigma_{i=1}^k (g_i'' + h_i'')$ is a part of the base locus of D.

Set $X_2^{(0)} = X_2 \backslash A$ the Zariski open set of X_2. We prove that the linear system $|D|$ separates points on $X_2^{(0)}$; i.e., For any points $R_1, R_2 \in X_2^{(0)}$, one can find $\phi \in H^0(X_2, D)$ such that $\phi(R_1) = 0$ and $\phi(R_2) \neq 0$. Put $R_i' = \pi_2(R_i)$ and $R_i'' = \delta\tau(R_i')$ for $i = 1, 2$.

1) When $R_1'' \neq R_2''$: If the condition (a) holds, then $K_C + D_m + D_n$ is a very ample divisor on C. Thus $|\tau^*\delta^*(K_C + D_m + D_n) + \Sigma g_i|$ separates R_1' and R_2'. Hence $|D|$ separates R_1 and R_2. If the condition (b) holds, then the system N^k separates R_1'' and R_2''. Hence $|D|$ also separates R_1 and R_2.

2) When $R_1' \neq R_2'$ and $R_1'' = R_2''$: Since $K_C + D_m + D_n \geq 0$ and $K_C + 2D_m + D_n - \widehat{P_1} - \cdots - \widehat{P_k} \geq 0$, the system $|\tau^*(C_0 + \delta^*(K_C + D_m + D_n) + \Sigma(g_i - h_i))|$ separates R_1' and R_2'. Hence $|D|$ separates R_1 and R_2.

3) When $R_1'' = R_2''$: We consider the restriction map

$$\lambda : H^0(X, D) \longrightarrow H^0(\pi_2^{-1}(R_1'), D|_{\pi_1^{-1}(R_1')}) \simeq \mathbf{C}^2.$$

By (13), the map λ is given by

$$\phi \longmapsto (\phi^{(0)}(R_1'), \phi^{(1)}(R_1')).$$

Thus by the same argument as above, λ is surjective. Hence $|D|$ separates R_1 and R_2.

By 1),2) and 3), $|D|$ separates points on $X_2^{(0)}$. Therefore $|K_{X_2} + S_2|$ induce a birational map on X_2, and this map is holomorphic on $X_2^{(0)}$. Since the generic point of S_2 is contained in $X_2^{(0)}$, $|\omega_{S_2}|$ also induces a birational map on S_2. Hence $\Phi_{K_{\tilde{S}}}$ is a birational map.

Next we prove that the base locus $Bs|\omega_{S_5}|$ is empty. By the above argument, we have $Bs|\omega_{S_5}| \subset \cup_i h_i'''$. Since $\omega_{S_5} = (\mu_5)_*(\widehat{\mu_4} \circ \widehat{\mu_3} \circ \widehat{\mu})^*\omega_S \otimes \mathcal{O}_{S_5}(-\Sigma_i h_i''')$, the divisor $\Sigma_i h_i'''$ is not a fixed component of $|\omega_{S_5}|$. For $1 \leq i \leq k$, let P_i', P_i'' be the points on h_i''' which are the image of two exceptional curves $(\widehat{\mu_4})^{-1}(\widehat{g_i})$ by μ_5. For a general member

$\psi \in M^k$, let C_ψ be the curve on $W \simeq T \subset X$ defined by ψ. Then the strict transform of C_ψ by $\mu_4 \circ \mu_3 \circ \mu$ does not intersect with $(\widehat{\mu_4})^{-1}(\widehat{g_i})$. Therefore the points P_i' and P_i'' $(1 \leq i \leq k)$ are not the base points of $|\omega_{S_5}|$. Thus $Bs|\omega_{S_5}|$ is empty. Hence $Bs|K_{\widetilde{S}}|$ is empty.

Especially $\widetilde{S}$ is relatively minimal, and $\Phi_{K_{\widetilde{S}}}$ is a holomorphic map.

q.e.d.

From Propositions *1.7* and *3.1*, we have the following by an easy computation :

Theorem 3.2 *Let x, y be any integers with the following :*
(a) If $g = 0$, then one of (a.1) $\sim$ (a.4) is satisfied ;
(a.1) $3x - 10 \leq y \leq 4x - 16$,
(a.2) $y = 4x - 15$, $x \geq 5$ *and* $x \equiv 1 \ (mod\, 2)$,
(a.3) $y = 4x - 14$ *and* $x \geq 5$,
(a.4) $y = 4x - i$, $i = 13$ *or* 12, $x \geq 6$ *and* $x \equiv 0 \ (mod\, 2)$.
(b) If $g \geq 1$, then one of (b.1) $\sim$ (b.5) is satisfied ;
(b.1) $3x + 10g - 10 \leq y \leq 4x + 8g - 18$,
(b.2) $y = 4x + 8g - 17$, $x \geq 2g + 8$ *and* $x \equiv 0 \ (mod\, 2)$,
(b.3) $y = 4x + 8g - 16$,
(b.4) $y = 4x + 8g - i$, $i = 15$ *or* 14, $x \geq 2g + 7$ *and* $x \equiv 0 \ (mod\, 2)$,
(b.5) $y = 4x + 8g - 12$, $x \geq 2g + 6$ *and* $x \equiv 0 \ (mod\, 2)$.
Then there exists a minimal algebraic surface S such that
1) $\chi(\mathcal{O}_S) = x$, $c_1^2(S) = y$ *and* $q(S) = g$,
2) the canonical map Φ_{K_S} is a birational holomorphic map,
3) S has a fibration of nonhyperelliptic curves of genus 3 over a curve of genus g.

From Theorem *3.2*, we have the following by an easy computation :

Theorem 3.3 *Let x, y be any integers with ;*
(a) $3x - 10 \leq y \leq 8x - 78$, *or*
(b) $y = 8x - i$ $(35 \leq i \leq 77,\ i \neq 39, 41, 47)$ *and x satisfies the condition in the following table ;*

i	35	36	37	38	40	42	43	44	45	46	48	49	50	51
x	5	E	6	6	E	D	D	A	7, 8	8	E	E	A	D

52	53	54	55	56	57	58	59	60	61	62	63
A	9, 10	D, 10	D	E	E	A	A	A	11, 12	D, 12	D

64	65	66	67	68	69	70	71	72	73	74	75	76	77
A	A	A	A	A	E, 13	A	D	A	A	A	A	A	E, 15

where "A" (resp. "E", resp. "D") means any (resp. any even, resp. any odd) integer with $\geq (i/5) - 2$. *(For instance, if* $y = 8x - 54$, *then* x *is any odd integer with* ≥ 9 *or* $x = 10$.*)*

Then there exists a minimal algebraic surface S *such that*

1) $\chi(\mathcal{O}_S) = x,\ c_1^2(S) = y$,

2) Φ_{K_S} *is a birational holomorphic map,*

3) S *has a fibration of nonhyperelliptic curves of genus 3.*

Remark 3.4 There exists a minimal algebraic surface S of general type such that

1) $c_1^2(S) = 4p_g(S) - 6,\ q(S) = 0,\ p_g(S) \geq 4$ and $p_g(S) \equiv 1\ (mod\, 2)$,

2) Φ_{K_S} is generically $4:1$ map,

3) S has a fibration $\eta : S \longrightarrow \mathbf{P}^1$ of nonhyperelliptic curves such that Φ_{K_S} is induced by a relativation of the generic projection of the plane quartic curve which is the general fiber of h.

In fact, in Proposition *1.7*, we set $m \geq 3,\ n = 0$ and $k = 2m - 1$. Then the assertion is verified by the similar argument as in the proof of Proposition *3.1*.

Remark 3.5 There exists a degeneration $\rho : \mathcal{S} \longrightarrow \Delta = \{|t| < 1\}$ of surfaces of general type such that the main component $S^{(0)}$ of the degenerate fiber $S_0 = \rho^{-1}(0)$ satisfies

$$p_g(S^{(0)}) = p_g(S_t) - 1,\ c_1^2(S^{(0)}) = c_1^2(S_t) - 2,\ q(S^{(0)}) = q(S_t),$$

where $S_t = \rho^{-1}(t)$ $(t \neq 0)$ is a smooth fiber of ρ. This degeneration is observed in the same region as in Theorem *3.2*. The construction is similar to [2, §4].

Other examples of "tame" degenerations can be seen in [9], [20], [21], [2], etc.

References

[1] T. Ashikaga and K. Konno: Algebraic surfaces of general type with $c_1^2 = 3p_g - 7$, preprint (submitted to Tôhoku Math. J.).

[2] T. Ashikaga and K. Konno: Examples of degenerations of Castelnuovo surfaces, preprint.

[3] W. Barth, C. Peters and A. Van de Ven, *Compact complex surfaces*, Ergeb. Math. Grenzgeb. (3) Band 4, Springer, 1984.

[4] G. Castelnuovo: Osservazioni intorno alla geometria sopra una superficie, Nota II. Rendiconti del R. Instituto Lombardo, s.II, vol 24, 1891.

[5] F. Catanese: On the moduli spaces of surfaces of general type, J. Differential Geom. 19 (1984), 483–515.

[6] F. Catanese: Canonical rings and "special" surfaces of general type, in *Proc. Symp. Pure Math.* vol 46 (1987), A.M.S., 175–194.

[7] Z. Chen: On the geography of surfaces, Math. Ann. 277 (1987), 141–164.

[8] Z. Chen: The existence of algebraic surfaces with preassigned Chern numbers, preprint.

[9] R. Friedman: A degenerating family of quintic surfaces with trivial monodromy, Duke Math. J. 50 (1983), 203–214.

[10] F. Hirzebruch: Hilbert modular surfaces, Enseign. Math. 19 (1973), 183–281.

[11] E. Horikawa: On deformation of quintic surfaces, Invent. Math. 31 (1975), 43–85.

[12] E. Horikawa: Algebraic surfaces of general type with small c_1^2, I, II, III, IV. Ann. Math. 104 (1976), 358–387, Invent. Math. 37 (1976), 121–155, ibid. 47 (1978), 209–248, ibid. 50 (1979), 103–128.

[13] K. Konno: Algebraic surfaces of general type with $c_1^2 = 3p_g - 6$, preprint.

[14] R. Miranda: Triple covers in algebraic geometry, Amer. J. Math. 107 (1985), 1123–1158.

[15] U. Persson: On Chern invariants of surfaces of general type, Compositio Math. 43 (1981), 3–58.

[16] U. Persson: An introduction to the geography of surfaces of general type, in *Proc. Symp. Pure Math.*, vol. 46 (1987), A.M.S., 195–218.

[17] M. Reid: Elliptic Gorenstein singularities of surfaces, preprint (1976 ?).

[18] K. Saito: Einfach-elliptishe singularitäten, Invent. Math. 23 (1974), 289–325.

[19] H. Tokunaga: On the construction of triple coverings of algebraic surfaces of certain type, preprint.

[20] S. Usui: Type I degeneration of Kunev surfaces, to appear in Astérisque, *Colloque Théorie de Hodge, Luminy 1987.*

[21] S. Usui: Mixed Torelli problem for Todorov surfaces, preprint.

[22] G. Xiao: *Surfaces fibrées en courbes de genre deux*, Lecture Notes in Math. 1137, Springer, 1985.

[23] G. Xiao: Algebraic surfaces with high canonical degree, Math. Ann. 274 (1986), 473–483.

[24] G. Xiao: Hyperelliptic surfaces of general type with $K^2 < 4\chi$, manuscripta math. 57 (1987), 125–148.

[25] H. Yoshihara: Plane curves whose singular points are cusps and triple coverings of $\mathbf{P}^2$, manuscripta math. 64 (1989), 169–187.

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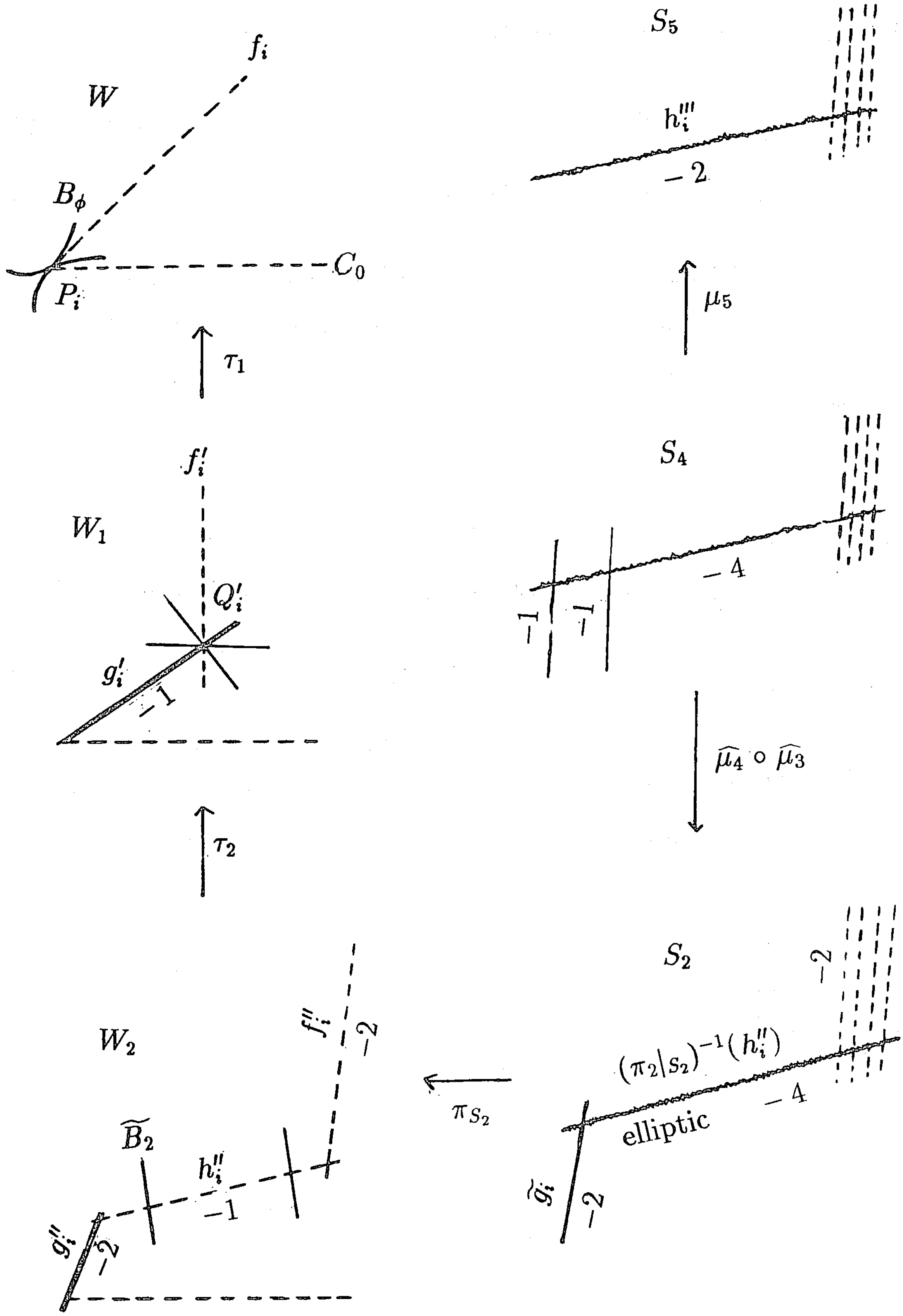
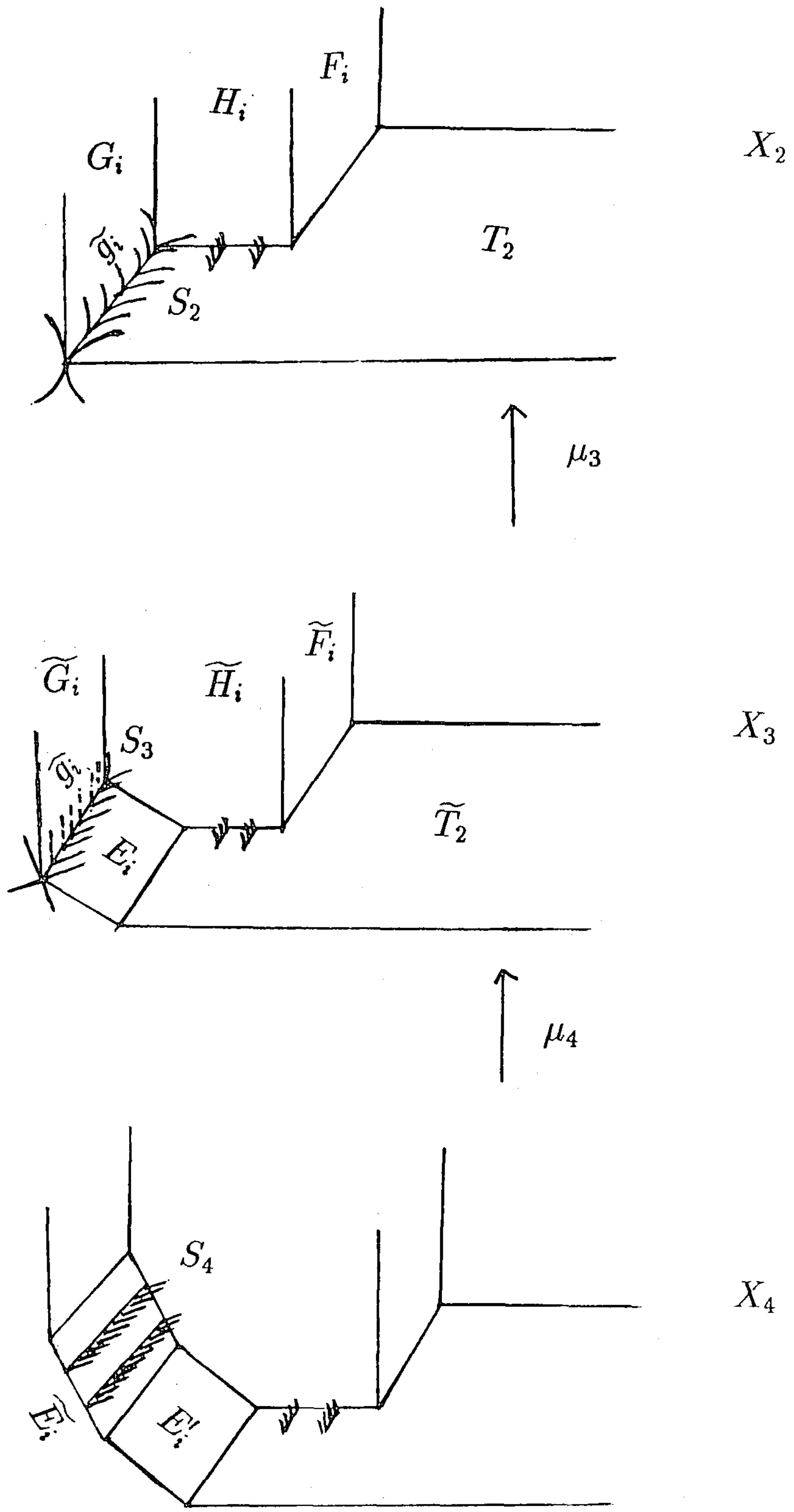
Fig. 1



W
f_i
B_ϕ
C_0
P_i
τ_1
W_1
f_i'
Q_i'
g_i'
-1
τ_2
W_2
f_i''
-2
$\widetilde{B_2}$
h_i''
-1
g_i''
-2
π_{S_2}
S_5
h_i'''
-2
μ_5
S_4
-4
-1
-1
$\widehat{\mu_4} \circ \widehat{\mu_3}$
S_2
-2
$(\pi_2|_{S_2})^{-1}(h_i'')$
-4
elliptic
$\widetilde{g_i}$
-2

Fig. 2



F_i
H_i
G_i
X_2
$\widetilde{g}_i$
T_2
S_2
μ_3
$\widetilde{F}_i$
$\widetilde{G}_i$
$\widetilde{H}_i$
X_3
S_3
$\widetilde{g}_i$
$\widetilde{T}_2$
E_i
μ_4
S_4
X_4
$\widetilde{E}_i$
E'_i

Appendix to T. Ashikaga's paper

KAZUHIRO KONNO

Introduction.

The purpose of this appendix is to give another construction of surfaces of general type with pencils of nonhyperelliptic curves of genus 3, whose canonical map is a birational morphism onto its image. We restricts ourselves to regular surfaces here, whereas our method can be applied to irregular ones as well (with some more effort).

Let S be a smooth surface with a morphism $f : S \to C$ onto a smooth curve C. Recall that, if a general fiber is a nonhyperelliptic curve of genus three, there is a canonical birational map of S into a $\mathbf{P}^2$-bundle W on C (see, [H, §1]). We let X be its image, and consider the fibration $g : X \to C$ induced by the projection of W. Then the differece of the invariants $(\chi(\mathcal{O}_X) - \chi(\mathcal{O}_S), \omega_X^2 - \omega_S^2)$ can be considered as the contribution of the singular fibers of g. Though we do not have a complete list of possible singular fibers, we at least can expect that they are quite similar to those in [K]. However, the singular fiber arising from a simple elliptic singularity of type $\tilde{E}_7$, which Ashikaga has constructed, seems to be a "special" one ([K, §9]). What this means may be garanteed by the fact that the canonical bundle of S cannot be ample in this case. Thus there should be a way to construct "general" ones. This is the motivation of the present note.

Construction.

We let W denote the total space of the $\mathbf{P}^2$-bundle

$$\pi : \mathbf{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)) \to \mathbf{P}^1,$$

where a, b, c are integers satisfying $0 \leq a \leq b \leq c$. We let T and F denote the relatively ample tautological divisor and a fiber of π, respectively. Then the Picard group of W is a free abelian group generated by them. Further, we have $T^3 = (a+b+c)T^2F$ in the Chow ring of W. We put

$$p = a+b+c+3 \tag{1.1}$$

and assume that $p \geq 4$. Let s be an integer, and let Q be a general member of the linear system $|2T+sF|$. We remark that Q is irreducible and has only rational double points of type A_1 if

$$a+c+s \geq 0, \quad 2b+s \geq 0. \tag{1.2}$$

Choose general k fibers $F_1, \cdots, F_k$ of π. We assume that Q and F_i meet transversally, and that the intersection $Q_i = Q \cap F_i$ is an irreducible conic (in $F_i \simeq \mathbf{P}^2$) for each i.

We let $\nu : \hat{W} \to W$ be the blowing-up along $\cup Q_i$, and put $\mathcal{E}_i = \nu^{-1}(Q_i)$. Since the normal sheaf of $Q_i \simeq \mathbf{P}^1$ in W is isomorphic to $\mathcal{O} \oplus \mathcal{O}(4)$, each $\mathcal{E}_i$ is isomorphic to Σ_4, the Hirzebruch surface of degree 4. On $\hat{W}$, we consider the linear system $|L|$, where

$$L = \nu^*(4T - (p-5-k)F) - 2\sum_{i=1}^{k} \mathcal{E}_i. \tag{1.3}$$

If we denote by $\hat{Q}$ and $\hat{F}_i$ the proper transforms of Q and F_i, respectively, then we get

$$\begin{aligned} L &\sim \nu^*(4T-(p-5+k)F) + 2\textstyle\sum_{i=1}^k \hat{F}_i \\ &\sim 2\hat{Q} + (k+5-p-2s)\nu^*F, \end{aligned}$$

where the symbol $\sim$ means the linear equivalence of divisors. Since $\hat{Q} \cap \hat{F}_i = \emptyset$, we have

Lemma 1: *$Bs|L| = \emptyset$ if the following conditions are satisfied.*

1) $k \geq 2s + p - 5$.

2) *Q does not meet $Bs|4T-(p-5+k)F|$.*

Lemma 2: *The condition* 2) *of* Lemma 1 *is satisfied if one of the following conditions is satisfied:*

1) $k \leq 4a - p + 5 = 3a - b - c + 2.$

2) $s = -2a, \ k \leq 4b - p + 5 = 3b - a - c + 2.$

Proof. If 1) holds, then $Bs|4T - (p - 5 + k)F| = \emptyset$. We assume that 2) holds. We let X_0, X_1 and X_2 be sections on W of $\mathcal{O}(T - aF)$, $\mathcal{O}(T-bF)$ and $\mathcal{O}(T-cF)$, respectively, such that they form a system of homogeneous coordinates on each fiber of $\pi : W \to \mathbf{P}^1$. If $4b \geq p-5+k$, then $|4T - (p - 5 + k)F|$ has no base point outside the rational curve B defined by $X_1 = X_2 = 0$. On the other hand, the equation of Q can be written as

$$q_{00}X_0^2 + q_{10}X_0X_1 + q_{01}X_0X_2 + q_{20}X_1^2 + q_{11}X_1X_2 + q_{02}X_2^2 = 0,$$

where q_{ij} is a homogeneous form on $\mathbf{P}^1$ of degree $(2-i-j)a+ib+jc+s$. Thus, if $s = -2a$, then we can assume that q_{00} is a nonzero constant. Then Q does not meet B. Thus 2) is also sufficient to imply 2) of Lemma 1. *q.e.d.*

Lemma 3: *Suppose that $|L|$ contains an irreducible nonsingular member S. Then S is a minimal surface satisfying:*

1) *The canonical map of S is a birational morphism.*

2) $p_g(S) = p$, $q(S) = 0$ *and* $c_1^2(S) = 3p_g(S) - 7 + k$, *where* p_g, q *and* c_1^2 *denote the geometric genus, the irregularity and the Chern number of* S, *respectively.*

Proof. For any complex manifold M, we denote by K_M the canonical line bundle of M. Then, by the adjunction formula, we have

$$K_S \sim (K_{\hat{W}} + S)|_S \sim (\nu^*T + \sum_{i=1}^{k} \hat{F}_i)|_S.$$

Since $\hat{F}_i$, $1 \leq i \leq k$, does not meet S, we have $\mathcal{O}(K_S) = \mathcal{O}_S(\nu^*T)$. We next consider the cohomology exact sequences derived from

$$0 \to \mathcal{O}(K_{\hat{W}}) \to \mathcal{O}(\nu^*T + \sum_{i=1}^{k} \hat{F}_i) \to \mathcal{O}(K_S) \to 0, \qquad (1.4)$$

$$0 \to \mathcal{O}(\nu^*T) \to \mathcal{O}(\nu^*T + \sum_{i=1}^{k} \hat{F}_i) \to \oplus_{i=1}^{k} \mathcal{O}_{\hat{F}_i}(-1) \to 0. \qquad (1.5)$$

Since $\hat{W}$ is rational, we have $H^q(\hat{W}, \mathcal{O}(K_{\hat{W}})) = 0$ for $q < 3$. Thus we get $H^q(S, \mathcal{O}(K_S)) \simeq H^q(\hat{W}, \mathcal{O}(\nu^*T + \sum \hat{F}_i))$ for $q < 2$ from (1.4). Then, by (1.5), we get $H^q(K_S) \simeq H^q(W, \mathcal{O}(T))$ for $q < 2$. This shows the formulae for $p_g(S)$ and $q(S)$. Note that we in particular have shown that $|\nu^*T|$ is restricted to $|K_S|$ isomorphically. Thus $Bs|K_S| = Bs|\nu^*T| = \emptyset$. This implies that S is minimal and we get 1). Finally, we calculate c_1^2 to get:

$$c_1^2(S) = (\nu^*T)^2 L = T^2(4T - (p - 5 - k)F) = 3p - 7 + k.$$

q.e.d.

Now, varying a, b, c, s and k under the conditions as above, we obtain a series of surfaces to show the following:

Proposition: *Let x and y be positive integers satisfying*

$$x \geq 4, \quad 3x - 7 \leq y \leq \begin{cases} 4x - 8 & \text{if } x \text{ is odd,} \\ 4x - 10 & \text{if } x \text{ is even.} \end{cases}$$

Then there exists a minimal, regular surface of general type S with the following properties:

1) $p_g(S) = x, \ c_1^2(S) = y.$

2) *The canonical linear system of S has neither fixed components nor base points, and the canonical map is a birational morphism onto its image.*

3) *S has a pencil of nonhyperelliptic curves of genus three.*

4) *The canonical image of S is contained in a threefold of minimal degree.*

Sketch of Proof. Put $(a, b, c) = (a, a, a)$, $(a, a, a + 1)$, $(a, a + 1, a + 1)$ according to $x \equiv 0, 1, 2$ modulo 3, respectively. Then, by 1) of Lemma 2, we can cover the region

$$3x - 7 \le y \le \frac{10}{3}x - \begin{cases} 6, & \text{if } x \equiv 0, \\ \frac{22}{3}, & \text{if } x \equiv 1, \\ \frac{26}{3}, & \text{if } x \equiv 2, \end{cases} \qquad (\text{mod.}3) \qquad (1.6)$$

for a suitably chosen s satisfying (1.2).

We next put $s = -2a$ and consider 2) of Lemma 2. Putting $a = 0$, $b = c$, we can cover the region $4x - 12 \le y \le 4x - 8$ with x odd. Similarly, putting $a = 1$, $b = c$, we can cover the region $4x - 16 \le y \le 4x - 10$ with x even. In this way, by increasing a and putting $b = c$, we can cover the region outside (1.6) as well. The other statements follow from Lemma 1.3 and the construction. *q.e.d.*

Remark: The inverse image of Q_i on S is a hyperelliptic curve of genus 3 (see, [K, §9]). Further, it can be checked that most surfaces we have constructed have ample canonical bundle.

References.

[K] K. Konno: Algebraic surfaces oi general type with $c_1^2 = 3p_g - 6$, preprint.

[H] E. Horikawa: On algebraic surfaces with pencils of curves of genus 2, In: *Complex Analysis and Algebraic Geometry*, a volume dedicated to K. Kodaira, pp. 79–90, Tokyo and Cambridge: Iwanami Shoten Publishers and Cambridge University Press, 1977.

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Smooth projective varieties dominated by homogeneous rational projective variety.

By Koji Cho and Eiichi Sato

In this paper we consider the following problem due to Remmert , Van de Ven [R,V] and Lazarsfeld [L].

Problem Let X be a smooth projective variety defined over the complex number field $\mathbf{C}$. Let $f : G / P \longrightarrow X$ be a surjective morphism where G is a simple classical group and P a maximal parabolic subgroup of G. Then unless f is an isomorphism, X is isomorphic to a projective space.

Here we study the property of such a variety X and have

Main Theorem Let X be a smooth projective variety, Q a smooth quadric hypersurface of $\dim \geq 3$ and f a surjective morphism defined over an algebraically closed field of any charactristic (char $k \neq 2$). Assume that f is separable. Then unless f is an isomorphism, X is isomorphic to a projective space.

Mori showed in [M] that a smooth projective variety with ample tangent bundle is isomorphic to a projective space. As an important application Lazarsfeld in [L] proved that if G / P is a projective space, the above problem is valid (even in any characteristic. Use Proposition 1.1 in his proof [L])

In our case a key for the proof of Main Theorem is to apply Mori's Theorem and to study the behavior of a extremal rational curve on X and the property about the tangent bundle T_Q of a quadric hypersurface Q.

Precisely speaking, in §1, we observe a general nature of the variety X dominated by G / P and see that $v^*T_X \simeq \mathcal{O}_{P^1}(2) \oplus \mathcal{O}_{P^1}(1)^{\oplus r} \oplus \mathcal{O}_{P^1}^{\oplus s}$ for the normalization $P^1 \longrightarrow C$ (= a general extremal rational curve in X) (Proposition 1.13)

In § 2, we study the property of the tangent bundle T_Q (Theorem 2.1), by which we obtain a result (Theorem 2.7) about the vector bundle F on Q which is a non-trivial extention of an ample line bundle on a hyperplane section of Q by T_Q.
Namely theorem 2.7 says that there is a line L on Q such that $F_{|L}$ is ample.

In §3, for an extremal rational curve C on X, we observe the possibility of a vector bundle v^*T_X with the normalization $v: P^1 \longrightarrow C$. Thus the case with the conditions below is left to us as an complicated one:

(1)the morphism $f: Q \longrightarrow X$ has the ramification locus R_Q which is a hyperplane section of Q with $f^{-1}f(R_Q) = R_Q$,

(2) for any line L on Q the restricted map $f_{|L} : L \longrightarrow f(L)$ is contained in one component H ($\ni [v]$) in $Hom(P^1,X)$

(3) the restriction of f^*T_X to any line in Q is not ample and finally

(4) f^*T_X is obtained as an extention of the line bundle $\mathcal{O}_{R_Q}(2)$ on R_Q by T_Q.

But the condition (3) (4) contradicts Theorem 2.7 and this case does not occur, which asserts Main Theorem.

After finishing the present paper we found that K.H.Paranjape and V.Srinivas got the same result (= Proposition 8) in the following paper:

Self maps of homogeneous aspcas.

Invent.math.98. 425-444(1989).

Our method of the proof is essencialy same as the one by them except for the last part of the proof. The last part seems to be most complicated one. That is why we have the argument in any characteristic ($\neq 2$). We must study the property of the homomorphism $T_Q \longrightarrow f^*T_X$ on the branched locus R_Q in details.

Notation. Basically we use custumary terminologies of algebraic geometry. A variety means a separated, reduced irreducible algebraic k-scheme where k is an algebraically closed field of any characteristic. $\mathcal{O}_{P^n}(1)$ denotes the line bundle corresponding to the hyperplanes in P^n and in case of $n=1$ it is abbriviated simply as $\mathcal{O}(1)$ very often. When a vector bundle E on a variety is generated by its global sections, for the simplicity we say that E is GS. For a variety X and a closed subscheme Y which is locally complete intersection in X, $N_{Y/X}$ means the normal bundle of Y in X.

§.1 General properties of a smooth target of G / P.

In this section we study the property of the smooth projective variety which is a target of homogeneous complete variety G / P.

Let us begin with easy propositions about vector bundle on a curve For the proof we use several facts in [H].

Proposition 1.1. (Lemma 4.5 in [L]) Let E, F be vector bundles on a smooth projective curve C with the same rank and $j: E \longrightarrow F$ an injective homomorphism. If E is an ample vector bundle, so is F.

Proof. Take an integer m such that $S^m(E)$ is ample, GS (= is generated by its global sections) and its first cohomology group vanishes. (see Proposition 2.4 and Proposition 3.3 [H]) Moreover consider the homomorphism $\bar{j} : S^m(E) \longrightarrow S^m(F)$ induced by $j: E \longrightarrow F$, which is also an injective homomorphism. Set T = Cokernel of $\bar{j}$. Since T is GS, so is $S^m(F)$. Now assume $S^m(F)$ is not ample, namely there is a section C' (with respect to the projection $P(S^m(F)) \longrightarrow C$) in $P(S^m(F))$ which maps to a point via the morphism induced by the tautological line bundle of $P(S^m(F))$. Thus we infer that there is a surjective homomorphism $S^m(F) \longrightarrow \mathcal{O}_C$, which gives rise to the non-trivial homomorphism $S^m(E) \longrightarrow \mathcal{O}_{C'}$. But since $S^m(E)$ is ample, we have a contradiction. Hence $S^m(F)$ is ample and therefore F is ample.

q.e.d.

Moreover we have

Proposition 1.2. Let E and F be vector bundles on P^1 and $j:E \longrightarrow F$ a generically surjective homomorphism. Then we have

1) If E is GS, so is F

2) If E is ample, so is F.

3) Particularly let $E = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}$ and rank $F = n$. Suppose that F is the pull back of a vector bundle G on P^1 via a finite morphism $f: P^1 \longrightarrow P^1$ and $\deg G \leq n + 1$. Then G is one of the following:

α) $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-1}$

β) $\mathcal{O}(3) \oplus \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}$.

γ) $\mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus n-3} \oplus \mathcal{O}$.

δ) $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}$.

Proof. Noting that any vector bundle on P^1 is a direct sum of line bundles, this proposition is trivial.

Now let us consider a separable, surjective morphism $f: Y \longrightarrow X$ betweem smooth projective varieties X, Y and let R_Y be the ramification divisor of Y via f. Then we have

Proposition 1.3. Under the above condition let Y be a Fano variety (i.e. $-K_Y$ is ample). Then we have

1) If f is finite and R_Y is nummerically effective i.e. the intersection number $R_Y C$ is non-negative for any integral curve C on Y, then X is a Fano variety.

2) Assume that Pic $Y \simeq Z$. Then the asssumptions in 1) hold good.

Proof. We have the equality: $K_Y = f^*K_X + R_Y$. Since the product of an ample line bundle and a nummerically effective line bundle is ample, $-K_X$ is ample by the finiteness of f. For 2) note that any effective divisor is ample. Unless f is finite there is a curve C in Y such that f(C) is a point. Take a divisor D in X which does not contain f(C). Since $f^{-1}(D)$ is ample, it intersects with the curve C which induces a contradiction.

q.e.d.

From now on let us consider

(1.4) a finite, separable and surjective morphism: $f:\ G / P\ (= W) \longrightarrow X$ where R_W is the branched locus on W with respect to the morphism f and $R_X = f(R_W)$.

Then it is well-known that W is a Fano variety and any effective divisor on W is nummerically effective. Hence by Proposition 1.3.1 X is a Fano variety.

Hereafter the property of the variety X is studied.

First let us treat the case that $R_W = \phi$. For the purpose we state this case in more general form.

Proposition 1.5. Let Z and U be smooth projective varieties and $f: U \longrightarrow Z$ an ètale finite morphism. Assume that $\chi(U, \mathcal{O}_U) = 1$. Then, f is an isomorphism.

Proof. The assumption says that $f^*T_Z = T_U$.
Thus, Hirzebruch Atiyah-Singer Riemann-Roch theorem implies that $\deg f \times \chi(Z, \mathcal{O}_Z) = \chi(U, \mathcal{O}_U) = 1$. Hence f is a isomorphism.

q.e.d.

Corollary 1.5.1. Any smooth projective Fano variety Z defined over the complex number field is algebraically simply connected.

Proof. Let $f: U \longrightarrow Z$ be a finite etale morphism from a algebraic scheme. Then we see that U is a smooth projective variety Since $f^*K_Z = K_U$, U is a Fano variety. Hence by virtue of Kodaira's vanishing Theorem, we get $H^i(Z,\mathcal{O}_Z) = 0$ for $1 \le i \le \dim Z - 1$. Thus, Proposition 1.5 asserts that f is an isomorphism.

q.e.d.

Now let us recall a characterization of of a uniruled variety.

Proposition 1.6. Let M be a smooth projective variety. Then, the following two conditions are equivalent to each other:

(1) M is separably uniruled namely, there is a separable surjective rational map from some projective space to M.

(2) there is a rational curve C in M such that for the normalisation $\varphi : \bar{C} \longrightarrow C$, φ^*T_X is generated by its global sections.

For the proof, for example, see Lemma 1.2 in [S].

Proposition 1.7. Under the notation and the condition in 1.4, let Z be an integral curve on X. Assume Z is not in R_X. Then for the normarization $g: \bar{Z} \longrightarrow Z$, g^*T_X is generically generated by its global sections. Particularly assume that Z is a rational curve. Then g^*T_X is GS.

Proof. Let $f^{-1}(Z) = Z_1 \cup \ldots \cup Z_r$ be the irreducible decomposition by the finiteness of f. Fix Z_1 and take the normalization $h:\bar{Z}_1 \longrightarrow Z_1$. Then there is a natural morphism $\bar{f}: \bar{Z}_1 \longrightarrow \bar{Z}$. . Now the morphism f induces the homomorphism: $T_{G/P} \longrightarrow f^*T_X$ which is generically surjective on Z_1. Since the tangent bundle of homogeneous space is generated by its global sections, so is $h^*T_{G/P}$. Thus we get the former. The latter is trivial.

q.e.d.

Combining Proposition 1.6 and Proposition 1.7, we get

Corollary 1.7.1. The variety X in 1.4 is separably uniruled.

(1.8) Thus let S be a set: {a rational curve C in $X \mid C \not\subset R_X$} and $c = \min\{(C. -K_X) \mid C \in S\}$. Now take a curve C in S with $c = C. -K_X$ and the normalisation of C: $\varphi: \bar{C} \longrightarrow C$. Then by Proposition 1.7, φ^*T_X is GS.

Let H be the irreducible component of the Hilbert scheme $\mathrm{Hom}(P^1, X)$ containing the point $[\varphi]$. (Note the $H^1(P^1, \varphi^*T_X) = 0$, therefore $\mathrm{Hom}(P^1, X)$ is smooth at $[\varphi]$)

Then we have an important

Lemma 1.8. Let H and R_X be as above. Then, for each element v in H with $v(P^1) \not\subset R_X$, $v: P^1 \longrightarrow v(P^1)$ is birational.

Proof. Assume that there is an element v in H such that $v: P^1 \longrightarrow v(P^1)$ is not birational. Then the morphism v factors to a finite morphism $h: P^1 \longrightarrow P^1$ (deg h ≥ 2) and a birational morphism $v_1: P^1 \longrightarrow X$. Then

$\deg v^*T_X = \deg h \times \deg v_1^*T_X > \deg v_1^*T_X = -v_1(P^1).K_X$

This contradicts the minimality of c.

q.e.d.

(1.9) Now let P be a smooth point of the image $\varphi(P^1)$ $(= C)$ (1.8) and let $\iota: o \longrightarrow P (\in X)$ a map with a point o in P^1. Then, it is known that $\mathrm{Hom}(P^1, X; \iota)$ is a closed subscheme in $\mathrm{Hom}(P^1, X)$ by Proposition 1 in [Mo].

Hence, letting $H_P = \{v \in H \mid v(o) = P\}$, we must remark that $H_P = H \cap \mathrm{Hom}(P^1, X; \iota)$ as a set.

Now we have

(1.9.1) Remark. 1) Let v be an element in H with $v(P^1) \not\subset R_X$. Then, v^*T_X is GS and hence H is smooth at $[v]$ because of the fact $H^1(P^1, v^*T_X \otimes \mathcal{O}_{P^1}(-1)) = 0$ by Proposition 1.7.

Hence if P is not in R_X, then H_P is smooth. In this case H_P may consist of finitely many components.

Hereafter till the end of this paper, we use the notations φ, H, P and H_P just above very often.

Now we shall state a sufficient condition for a rational curve in a projective variety to deform to a union of rational curves (which may not be distinct) which is implicitely shown in Theorem 4 in [Mo].

(1.10) Proposition (Theorem 4 in [Mo]) Let M be a non-singular projective variety, U a variety and $g: P^1 \times U \longrightarrow M$ a morphism satisfying the following properties:

1) for each point u in U, $g(P^1, u)$ $(= C_u)$ is a curve in M and for

each pair (u_1, u_2) in $U \times U$ $(u_1 \neq u_2)$, $C_{u_1} \neq C_{u_2}$.

2) $\dim g(P^1 \times U) \leq \dim U$ (≥ 2).

3) for each point u in U, there is a point m in M such that C_u passes through a point m and is smooth at m.

Then C_u deforms to a cycle which is a sum of rational curves (which may not be distinct). Precisely speaking, there is an open subset $\bar{Z}$ in $g(P^1 \times U)$(this is a constructible subset in M) satisfying the following:

1) $\bar{Z}$ does not contain the point m.

2) for every point z in $\bar{Z}$, there exists a cycle $\bar{C}$ as below to which some subfamily of $\{C_u\}$ deforms :

letting $\bar{C} = \bigcup_{i=1}^{k} a_i C_i$ where C_i is an irreducible component of C we have two cases:

1) there are two components C_i, C_j of $\bar{C}$ with $m \in C_i$ and $z \in C_j$.

2) there is a component C_i containing two points m, z with $a_i \geq 2$.

Proof. The properties 1) 2) say that for a general point n in $g(P^1 \times U)$ there is a closed curve $V(\subset U)$ so that every curve C_u $(u \in V)$ passes through the point n. Thus the proof of Theorem 4 in [Mo] implies the desired fact.

q.e.d.

(1.11) Let M be a smooth projective variety and C a rational curve in M. Let $\phi: \bar{C} \longrightarrow C$ be the normalisation of C. Assume that $\phi^* T_M = \bigoplus \mathcal{O}_{P^1}(a_i)$ with $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$. Moreover let $\iota: o \longrightarrow m$ be a map from a point o in P^1 to a point m $(\in C)$ in M where C is smooth at the point m and let us consider an irreducible component N of $\mathrm{Hom}(P^1, M; \iota)$ containing the morphism ϕ. For a point v in H, let $v^* T_M = \oplus \mathcal{O}(a_i^v)$ with $a_1^v \geq a_2^v \geq \ldots \geq a_n^v$.

Then Proposition 1.10 provides us with a

Proposition 1.12 Let the notation and condition be as in 1.11. Assume that
there is an open subset N_o in N enjoying three conditions 1) 2) 3) or 1) 2) 3')

1) φ is contained in N_o.

For every point v in N_o,

2) $a_n^v \geq 0$

3) $a_1^v + a_2^v \geq 4$.

3') $a_1^v \geq 3$ and $a_2^v = 0$.

(note that for every point v in N, $a_1^v \geq 2$.)

Finally suppose that the characteristic of the base field is zero.

Then C deforms to a cycle which is a sum of rational curves.

Proof. Set as $r + 1$ $\min\{i | a_i^v = 0$ for any $v \in N_0\}$.
By the semi-continuity of coherent sheaf, there is an open subset N' of N_0 such that $a_r^v > a_{r+1}^v = 0$ for any $v \in N'$.
Let W ($\subset M \times P^1 \times N'$) be the universal scheme of N' with the first projection $p: W \longrightarrow M$ and the third projection $q: W \longrightarrow N'$.
Now take a general point P in M and let D be an component of $qp^{-1}(P)$ in N'.
Since the characteristic of the base field is zero, the closure of $pq^{-1}(D)$ is of r dimension by virtue of Proposition 1.11 in [S].

On the other hand for such a point $v \in N'$, $\dim H^0(P^1, v^*T_M \otimes \mathcal{O}(-1))$ $= \sum_{i=1}^{r} a_i^v \geq r + 2$ and $\mathrm{Aut}(P^1, o)$ $(= \{\sigma \in \mathrm{Aut}\, P^1 | \sigma(o) = o\}$ is of 2-dimension, Thus we can construct a subscheme U in D and a morphism $g\colon P^1 \times U \longrightarrow X$ enjoying the three properties in Proposition 1.10. Hence we are done.

q.e.d.

Thus we get

Proposition 1.13. Let the notations R_X, H and H_p be as in 1.4 1.8 and 1.9. Assume that the characteristic of the ground field is zero. Then there is a Zariski open subset H^0 of H such that for each point u in $X - R_X$ and every point v in $H^0 \cap H_u$ v^*T_X is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus r} \oplus \mathcal{O}^{\oplus s}$.

Proof. The above corollary asserts that there is a point P outsides the subset R_X enjoying the following condition: there is a birational morphism $\varphi\colon P^1 \longrightarrow C \subset X$ so that $P \in C$ and v^*T_X is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus r} \oplus \mathcal{O}^{\oplus s}$.

Let W ($\subset X \times H \times P^1$) be the universal scheme of H with the first projection $p\colon W \longrightarrow X$, the second projection $q\colon W \longrightarrow H$ and $r\colon W \longrightarrow P^1$ the third projection.
Then since q is proper and $p^{-1}R_X$ is a closed subset in W, $qp^{-1}R_X$ is a properly closed set in H. Now set $A = \{v \in H | q^{-1}(v) \subset p^{-1}R_X\}$. Then we infer that A is closed in H. Letting $H_0 = H - A$, for any v in H_0, v^*T_X is GS.
Remark that the projection $q\colon W \longrightarrow H$ is P^1-bundle, therefore ,flat and $p^*\Omega_{X|q^{-1}(v)} \simeq v^*\Omega_X$.

Now let s be $\max\{h^0(P^1, v^*\Omega_X) \mid v \in H_0\}$ and $H^0 = \{v \in H_0 \mid h^0(P^1, \Omega_X) = s\}$. Then we infer that H^0 is open in H_0. Next doing the same procedure for the vector bundle $p^*\Omega_X \otimes r^*\mathcal{O}_{P^1}(1)$ on the P^1-bundle $q^{-1}(H^0) \longrightarrow H^0$, we complete the proof.

q.e.d.

Finally we finish this section to state a proposition which is implicitely shown in [Mo].

(1.14) Let $\bar{H}_P$ be an irreducible component of the above H_P. Herefter we follow the notations written in §3 of [Mo].

Let $G = \{g \in \mathrm{Aut}\, P^1 \mid g(o) = o\}$. Since the natural action of G on $\mathrm{Hom}_k(P^1, X; \iota)$ induces the action σ of G on the connected component $\bar{H}_P$:

$$\sigma : G \times \bar{H}_P \longrightarrow \bar{H}_P, \quad \sigma(g,v)x = v(g^{-1}x), \quad g \in G,\ v \in V,\ x \in P^1,$$

G also acts on $\bar{H}_P \times P^1$:

$$\tau: G \times \bar{H}_P \times P^1 \longrightarrow \bar{H}_P \times P^1, \quad \tau(g,v,x) = (\sigma(g,v), gx).$$

Let $\mathrm{Chow}^d X$ be the Chow variety parameterising 1-dimensional effective cycles C of X with $C.\,K_X^{-1} = d$. Then by Lemma 1.8, we have a morphism $\alpha: \bar{H}_P \longrightarrow \mathrm{Chow}^m X$ with $v(P^1).\,K_X^{-1} = m$ $(v \in \bar{H}_P)$.

Then Mori proved in [Mo]

Proposition 1.14. (Lemma 9 [Mo])

1) σ is free action. 2) (Y, Γ) is the geometric quotient of $\bar{H}_P$ by in the sense of [Mu] where Y is the normalization of the closure of $\Gamma(Y)$ in $\mathrm{Chow}^m X$.

§.2 The property of the tangent bundle of an n-dimensional smooth quadric hypersurface Q $(n \geq 3)$.

In this section we study the property of the tangent bundle T_Q of a non-singular quadric hypersurface in P^{n+1} (= P).

What we want to state here is

Theorem 2.1. 1) T_Q is GS. For each line L in Q, the restriction T_Q to L is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}$.
2) Let C be an integral curve in Q. If $T_{Q|C}$ is not ample, then C is a line in Q.

To show the above theorem, we make several reviews about Grassmann variety and Flag variety (see [OSS])

For an (n+1)-dimensional projective space P^{n+1}(= P), let G be a Grassmann variety parameterising lines in P and a flag variety F $\{(x,y) \in P \times G \mid x \in L_y\}$ where L_y is the line corresponding to a point y in G . Then we have canonical projections $p: F \longrightarrow P$, $q: F \longrightarrow G$ where p is a P^n-bundle and q a P^1-bundle.

Now the following is known.

Proposition 2.2. $F \simeq P(\Omega_P(2))$. The morphism q is a morphism induced naturally by the complete linear system of the tautological line bundle of $\Omega_P(2)$).(Of course, $\Omega_P(2)$ is GS). Moreover the restriction of the morphism q on each fiber of p ($\simeq P^n$) is an embedding.

Then we can show an important proposition which seems to be known. Here for a vector bundle E on Q, $E(m)$ denotes $E \otimes (\mathcal{O}_P(m)_{|Q})$.

Proposition 2.3. $T_Q \simeq \Omega_Q(2)$.

Proof. First a smooth quadric hypersurface Q can be described as G_{n+2} / Pa in P^{n+1} where $G_{n+2} = \{M \in SL(n+2,k) \mid {}^tMM = E\}$ is the orthogonal group and Pa is a maximal Parabolic subgroup of G_{n+2}.

Then $V(T_Q(-1))$ is isomorphic to $G_{n+2} \times k^n$ modulo the following relation:

let R be the radical part of Pa and φ :Pa $\longrightarrow GL(n,k)$ be the homomorphism where φ is trivial on R and the induced homomorphism $\bar{\varphi}$ yields the irreducible representation (Pa / R ($\simeq G_n$) $,k^n$) which is a canonical injective homomorphism $G_n \longrightarrow GL(n,k)$. Then the relation $(c,v) \sim (d,w)$ means that $c\,d^{-1} \in$ Pa and $v = \varphi(p)w$ with some point p in Pa.

On the other hand $V(\Omega_Q(1))$ is isomorphic to $G_{n+2} \times k^n$ modulo the relation obtained by replacing the homomorphism ${}^t\varphi^{-1}$ for the homomorphism φ defined as above. Remark that the property of the orthogonal group gives rise to the fact that ${}^t\varphi^{-1} = \varphi$ (defined as above). Thus we complete our proof.

q.e.d.

Thus an embedding $i: Q \longrightarrow P^{n+1}$ $(= P)$ naturally induces an important exact sequence:

$$(2.4) \qquad 0 \longrightarrow \mathcal{O}_Q \longrightarrow \Omega_P(2)_{|Q} \longrightarrow \Omega_Q(2) \; (= T_Q) \dashrightarrow 0$$

Hence we have a diagram:

$$
(2.4.1)\qquad
\begin{tikzcd}
 & & F = P(\Omega_P(2)) \arrow[dddll, "p"'] \arrow[ddddrr, "q"] \arrow[d, phantom, "\bigcup"] & & \\
 & & P(\Omega_P(2)_{|Q}) \arrow[d, phantom, "\bigcup"] & & \\
 & & P(\Omega_Q(2)) \arrow[dddl, "p'"'] \arrow[d, phantom, "\bigcup"] & & \\
 & & F(Q,1,0) \arrow[ddl, "\bar{p}"] \arrow[ddr, "\bar{q}"] & & \\
 & & & & \\
P^{n+1}\ (= P) \supset & Q & & G(Q) \subset & G
\end{tikzcd}
$$

where $G(Q)$ denotes the set of lines in Q and $F(Q,1,0)$ denotes the set $\{(x,y) \in Q \times G(Q) \mid x \in L_y$ (= the line corresponding to a point x)$\}$ ($\underset{\text{canonically}}{\subset}$ F). Then, a canonical projections $\bar{p}$: $F(Q,1,0) \longrightarrow Q$ is a Q^{n-3} (= $(n-3)$- dimensional smooth quadric hypersurface) -bundle and the other canonical projection $\bar{q}$: $F(Q,1,0) \longrightarrow G(Q)$ is a P^1-bundle.

Proof of Theorem 2.1.

The former of 1) is obvious since Q is a homogenous space. Next a line L of a smooth quadric surface Q_2 induces an exact sequence:

$$0 \longrightarrow T_L\ (= \mathcal{O}_{P^1}(2)) \longrightarrow T_{Q_2|L} \longrightarrow N_{L\,/\,Q_2}\ (=\mathcal{O}) \longrightarrow 0.$$

Thus we see $T_{Q_2|L} \simeq \mathcal{O} \oplus \mathcal{O}_{P^1}(2))$. Letting $\bar{Q}$ a smooth hyperplane section of Q, we get an exact sequence:

$$0 \longrightarrow T_{\bar{Q}} \longrightarrow T_{Q|\bar{Q}} \longrightarrow N_{\bar{Q}\,/\,Q}\ (=\mathcal{O}_{\bar{Q}}(1)) \longrightarrow 0.$$

Hence, the induction on the dimension of Q gives the latter of 1).

Now assume there is a curve C such that $T_{Q|C}$ is not ample. Then by the exact sequence 2.4,

$\Omega_P(2)_{|C}$ is not ample. On the other hand, we have

$$\mathcal{O}_{P(\Omega_P(2))}(1)_{|p^{-1}(C)} \simeq \mathcal{O}_{P(\Omega_P(2)_{|C})}(1).$$

Note that if a linear system of an ample line bundle has no base point, its induced morphism is finite. Thus the assumption $T_{Q|C}$ is not ample and Proposition 2.2 give rise to a section D (with respect to the p) in $P(\Omega_P(2))_{|C}$ such that q(D) is a point in G. Thus the diagram show that D is some fiber of q and therefore p(D) is a line in P (in Q).

q.e.d.

Let us consider an exact sequence of coherent sheaves on the smooth quadric hypersurface Q which is used in the final part in §3.

(2.5) $0 \longrightarrow T_Q \xrightarrow{h} F \longrightarrow G \longrightarrow 0$

where F is a torsion free sheaf on Q (dim Q $\geq$ 3) and G is a line bundle on an irreducible hyperplane section of Q (= R possibly singular).

Remark 2.5.1. if dim Q $\geq$ 4 or dim Q = 3 and R is singular, then Pic R $\simeq$ Z.

Remark.2.6.1. Assume G is GS. Then since $H^1(Q, T_Q) = 0$, F is GS.

Remark 2.6.2. Let b; Q $\longrightarrow$ P^n be a separable double covering with the branched locus B. Then B is a smooth hyperplane section of Q and there is the following exact sequence:

$$0 \longrightarrow T_Q \longrightarrow b^*T_{P^n} \longrightarrow \mathcal{O}_Q(2)_{|B} \longrightarrow 0.$$

Remark 2.6.3. Under the exact sequence 2.5 and $G = \mathcal{O}_Q(2)_{|R}$

$$\begin{aligned} \mathrm{Ext}^1(\mathcal{O}_Q(a)_{|R}, T_Q) &\simeq H^0(Q,T_Q) && (a = 1) \\ &\simeq k && (a = 2) \\ &= 0 && (a \geq 3) \end{aligned}$$

Proof. There is an exact sequence on Q:

$$0 \longrightarrow \mathcal{O}_Q(a-1) \longrightarrow \mathcal{O}_Q(a) \longrightarrow \mathcal{O}_Q(a)_{|R} \longrightarrow 0,$$

which induces the long exact sequence of cohomologies

$$\rightarrow H^0(Q,T_Q(-a+1)) \longrightarrow \mathrm{Ext}^1(\mathcal{O}_Q(a)_{|R},T_Q) \longrightarrow H^1(Q,T_Q(-a)) \longrightarrow H^1(Q,T_Q(-a+1))$$

On the other hand there are two exact sequences:

(2.6.4) $\quad 0 \longrightarrow T_Q(b) \longrightarrow T_P(b)_{|Q} \longrightarrow \mathcal{O}_Q(b+2) \longrightarrow 0,$

(2.6.5) $\quad 0 \longrightarrow T_P(b-2) \longrightarrow T_P(b) \longrightarrow T_P(b)_{|Q} \longrightarrow 0$

where Q is a smooth quadric hypersurface in the projective space P. If $\dim P \geq 4$, $H^i(P,T_P(b)) = 0$ for any integer b and $i = 1,2$ and therefore $H^1(Q,T_P(b)_{|Q}) = 0$. Thus, we get

$$H^0(Q,T_P(b)_{|Q}) \longrightarrow H^0(Q,\mathcal{O}_Q(b+2)) \longrightarrow H^1(Q,T_Q(b)) \longrightarrow 0.$$

Noting that $H^i(Q,T_Q(-1)) = 0$ for $i = 0,1$, $H^0(Q,T_Q(-2)) = 0$ and $\dim H^1(Q,T_Q(-2)) = 1$, we get the conclusion.

q.e.d.

Next we show a theorem which is important for the proof of Main Theorem.

Theorem 2.7 Under the condition 2.5, let us assume that F is locally free and the line bundle G is ample. Then there is a line L on Q where $F_{|L}$ is ample.

Proof. Assume that the conclusion is not true.

(2.7.1) for each line L on Q, $F_{|L}$ has a trivial line bundle as a direct summand. (see Remark 2.6.1)

Note that for every line L on Q, there is a unique surjective homomorphism $t_L: T_{Q|L} \longrightarrow \mathcal{O}_L \longrightarrow 0$ by Theorem 2.1.1 and $\operatorname{Ker} t_L$ is an ample vector bundle.

Now we have a

Claim 2.7.2. For every line L on Q, $F_{|L}$ has a unique trivial line bundle as a direct summand. Letting $f_L : F_{|L} \longrightarrow \mathcal{O}_L \longrightarrow 0$ a unique surjective homomorphism, $\operatorname{Ker} f_L$ is an ample vector bundle and there is the following diagram with an isomorphism g_L of $\mathcal{O}_L$:

$$\begin{array}{ccc} T_{Q|L} & \xrightarrow{h_{|L}} & F_{|L} \\ \downarrow{\scriptstyle t_L} & & \downarrow{\scriptstyle f_L} \\ \mathcal{O}_L & \xrightarrow[g_L]{} & \mathcal{O}_L \end{array}$$

Proof. In case of $L \not\subset R$, the restriction of the exact sequence 2.5 to the line L yields a generically surjective homomorphism $h_{|L}: T_{Q|L} \longrightarrow F_{|L}$. Since $T_{Q|L}$ has one trivial line bundle as a direct summand by Theorem 2.1.1, $F_{|L}$ has one trivial line bundle as a direct summand by assumption 2.7.1. Hence since $\operatorname{Ker} f_L$ is ample, $f_L\, h_{|L}(\operatorname{Ker} t_L) = 0$. Consequently $h|_L$ yields the isomorphism g_L of $\mathcal{O}_L$ as disired.

Next consider the case that $L \subset R$.

Restricting the above exact sequence 2.5 on R, we get two exact sequences:

(2.7.3) $0 \longrightarrow H \longrightarrow T_{Q|R} \xrightarrow{j} E \longrightarrow 0$

(2.7.4) $0 \longrightarrow E \xrightarrow{k} F_{|R} \longrightarrow G \longrightarrow 0$

where H is a linebundle on R and E a vector bundle of rank $n-1$ on R.

Note that $E_{|L}$ is GS and has at most one trivial line bundle as a direct summand by Theorem 2.1.1. Since the extension of ample vector bundle by an ample vector bundle is ample, $E_{|L}$ has one trivial line bundle as a direct summand by 2.6.1 and its quotient bundle is ample, which yields the desired result.

q.e.d.

Hereafter we use the notations in the diagram 2.4.1.

Now there is a canonical exact sequences of vector bundle on $P(T_Q)$:

(2.7.5) $\quad 0 \longrightarrow \operatorname{Ker} u \;(= K_1) \longrightarrow p'^{*}T_Q \xrightarrow{u} I_Q \longrightarrow 0,$

where I_Q is the tautological line bundle of T_Q.

Now we have a natural homomorphism:

(2.7.6) $\bar{q}^{*}\, \bar{q}_{*}\bar{p}^{*}\, F^{\vee} \xrightarrow{j} \bar{p}^{*}F^{\vee}$. Then claim 2.7.2 and the base change theorem say that

$\bar{q}_{*}\bar{p}^{*}\, F^{\vee}$ is a line bundle on $G(Q)$ and j in (2.7.6)

is an injective homomorphism as a vector bundle.

Here dualizing the sequence (2.7.6), we get

(2.7.7) $\quad 0 \longrightarrow \operatorname{Ker} w (= K_2) \longrightarrow \bar{p}^{*}F \xrightarrow{w} \bar{q}^{*}(\bar{q}_{*}\bar{p}^{*}\, F^{\vee})^{\vee} (= I_F) \longrightarrow 0$

on $F(Q,1,0)$.

(2.7.8) Note that $\bar{I}_Q$ and I_F are trivial on each fiber ($\simeq P^1$) of $\bar{q}$: $F(Q,1,0) \longrightarrow G(Q)$ in the diagram (2.4)

and K_1 and K_2 are ample vector bundles on it.

Moreover restrict the exact sequenace 2.7.5 to $F(Q,1,0)$ and consider the homomorphism $\bar{p}^{*}(h)$: $\bar{p}^{*}T_Q \longrightarrow \bar{p}^{*}F$ induced by the homomorphism $h: T_Q \longrightarrow F$. Now let us study the following diagram on each fiber ℓ of $\bar{q}$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \bar{K}_{1|\ell} & \xrightarrow{u'} & \bar{p}^* T_{Q|\ell} & \xrightarrow{\bar{u}} & \bar{I}_{Q|\ell} & \longrightarrow & 0, \\
 & & & & \downarrow \bar{p}^*(h) & & & & \\
0 & \longrightarrow & K_{2|\ell} & \xrightarrow{w'} & \bar{p}^* F_{|\ell} & \xrightarrow{w} & I_{F|\ell} & \longrightarrow & 0
\end{array}
$$

where $\bar{K}_1$ and $\bar{I}_Q$ are the restriction of two vector bundles K_1 and I_Q on $P(T_Q)$ to $F(Q,1,0)$ respectively.

Then (2.7.8) implies that the composition $u'\ \bar{p}^*(h)\ w$ of homomorphisms is zero map. Consequently the homomorphism $\bar{p}^*(h)$ yields the homomorphism $\bar{K}_1 \longrightarrow K_2$ and $\bar{I}_Q \xrightarrow{k} I_F$ canonically. Thus we see that k is an isomorphism by claim 2.7.2.

Now note that

$$\mathcal{O}_{P(T_Q)}(F(Q,1,0)) \simeq I_Q^{\otimes 2} \otimes p^* M$$

where M is a line bundle on Q.

The divisor $F(Q,1,0)$ in $P(T_Q)$ yields an exact sequence:

$$0 \longrightarrow \mathcal{O}_{P(T_Q)}(-F(Q,1,0)) \longrightarrow \mathcal{O}_{P(T_Q)} \longrightarrow \mathcal{O}_{F(Q,1,0)} \longrightarrow 0$$

Tensoring the tautological line bundle I_Q to the above exact sequence and taking the direct image $R^i p'_*$, we see that

$$T_Q \simeq \bar{p}_* \bar{I}_Q .$$

Moreover remarking that $\bar{p}_* \mathcal{O}_{F(Q,1,0)} \simeq \mathcal{O}_Q$ and taking the direct image $\bar{p}_*$ of the isomorphism $k: \bar{I}_Q \simeq I_F$,
we infer that $T_Q \simeq F$. This is absurd.

Hence we could prove Theorem 2.7.

q.e.d.

In case that R is singular and $G = \mathcal{O}_Q(a)_{|R}$ with $a \leq 2$, the author do not know whether a non-trivial extension class F in the exact sequence 2.5 is locally free or not.

But we can show

Corollary 2.8. Under the condition 2.5, assume that R is smooth and $G = \mathcal{O}_Q(2)|_R$. Then, there is a separable double covering $h;\ Q \longrightarrow P^n$ with the branched locus R and $F \simeq h^* T_{P^n}$.

Proof. Remark 2.6.2 and 2.6.3 assert this corollary.

§.3. The proof of Main Theorem.

In this section we show Main Theorem. ($\dim X \geq 3$)

(3.1) Let $f\colon Q \longrightarrow X$ be a finite, separable surjective morphism from a smooth quadric hypersurface Q to a smooth projective variety X and R_Q, R_X as in 1.4. Moreover let C be a rational curve ($\not\subset R_X$) in X which has the minimal degree with respect to $-K_X$ as was stated in 1.8 (see Corollary 1.7.1). We maintain the notations φ, P, H, H_P in 1.8 and 1.9.1 under the case of $W = Q$.

Now we have an important

Proposition 3.1. Let R_X, H be as above. Then we have

1) $\deg v^*T_X$ is independent of a choice of a point in H and the value is n or $n+1$.

2) Let v be an element in H such that $v(P^1) \not\subset R_X$.

If $\deg v^*T_X = n+1$, v^*T_X is one of the following:

α) $\mathcal{O}(2) \oplus \mathcal{O}(1)^{+n-1}$

β) $\mathcal{O}(3) \oplus \mathcal{O}(1)^{+n-2} \oplus \mathcal{O}$.

γ) $\mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(1)^{+n-3} \oplus \mathcal{O}$.

If $\deg v^*T_X = n$, v^*T_X is isomorphic to

δ) $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}$.

Proof. It is obvious that $\deg v^*T_X$ is constant for any v. Moreover Theorem 4 in [Mo] yields $\deg v^*T_X \leq n+1$. Next let us consider 2). First assume that all the component's of $f^{-1}(v(P^1))$ are lines. Then Theorem 2.1.1 and Proposition 1.2 apply. In case that C is a component of $f^{-1}(v(P^1))$ which is not a line, Proposition 1.2 (3) yields the desired fact.

q.e.d.

Hereafter, when v^*T_X is isomorphic to one (= ∓) of α, β, γ and δ in Proposition 3.1, v is said to of ∓ -type.

In the proof of Proposition 3.1, the following is shown

Corollary 3.2. Let v be an element of H with $v(P^1) \not\subset R_X$ Assume that v^*T_X is one of the three typies : β, γ and δ. Then $f^{-1}(v(P^1))$ is a union of lines in Q.

(3.3) Let v be as in Corollary 3.2 and L_v a line in Q with $f(L_v) = v(P^1)$. We give a sufficient condition for $f_{|L} : L \longrightarrow v(P^1)$ to be birational.

Proposition 3.4. Under the notations 3.1 and 3,3, assume that there is a point P ($\in R_X$) and a subset H' in H_P consisting of an infinite G-orbit where each element v in H' is of β , γ or δ type and G (= $\{g \in \mathrm{Aut}\, P^1 | g(o) = o\}$ in (1.11)) acts canonically on H_P.

Then for every line L on Q with $f(L) \not\subset R_X$, the restriction map $f_{|L}: L \longrightarrow f(L)$ is birational. Moreover the intersection number $L.\, f^*K_X$ is independent of a choice of a line L on Q and the value is $-n$ or $-n-1$.

Proof. First we show

Claim. Assume that there is an element v in H' and an irreducible component L_0 of $f^{-1}(v(L_0))$ (by corollary 3.2) so that L_0 is a line and $f_{|L_0}\ L_0 \longrightarrow f(L_0)$ is birational. Then for every line L with $f(L) \not\subset R_X$ $f_{|L}$ $L \longrightarrow f(L)$ is birational.

Proof. For every line L in Q, $(-f^*K_X . L) =$ $(\deg f_{|L})(f(L).-K_X)$ and if $f(L) \not\subset R_X$, $f(L).-K_X = n$ or $n+1$ by virtue of Proposition 3.1. Hence by the assumption we see that $-f^*K_X . L_0 = n$ or $n+1$ and therefore for every line L in Q with $f(L) \not\subset R_X$ $-f^*K_X . L = n$ or $n+1$ by means of Proposition 3.1 again. This give us the desired result.

Now let us return the proof of our propostion. Let $f^{-1}(P) = \{q_1,\ldots,q_r\}$ and $S = \{L_s |$ there is v in H'_P where L_s is a component of $f^{-1}(v(P^1))$ (corollary 3.2). Noting that S is an infinite set and a line in Q is determined by two points, we see that a general line $\bar{L}$ in S contains only one point q_i in $f^{-1}(P)$. Since $p \notin P_X$, f is umramified at q_i, which implies that the restricted morphism: $f_{\bar{L}}$ of f to such a line $\bar{L}$ is birational. The latter is proved in Claim.

q.e.d.

Remark 3.4.1. For a point q in Q, let C(q) be a set: {a line L in Q| $q \in L$}. Then if $n = \dim Q \geq 3$, C(q) is an (n-2) dimensional smooth quadric hypersurface in $P(T_{Q,q}{}^*)$ ($\simeq P^{n-1}$).

Corollary 3.4.2. Let the notations and the assumption be as in Proposition 3.4. Then there are an irreducible component $H_{P,0}$ in H_P and its divisor C_P is canonically induced by

$f(C(q))$ with some point P $(= f(q))$.

Proof. Proposition 3.4 says that there is an element v in the irreducible component $H_{P,0}$ in H_P, a line L on Q and a point q on L such that $f(L) = v(P^1)$, $f(q) = P$ and $f_{|L}$ is birational. Since H_P is an $(n+1)$-dimensional smooth variety, $C(q)$ induces the closed subscheme in H_P by Proposition 3.4. Hence our corollary follows from the fact that $\dim C_P = n$ by Remark 3.4.1 and that f is a finite morphism.

q.e.d.

Hence we have

Theorem 3.5. Let X be a smooth projective variety and $f: Q \longrightarrow X$ a separable surjective morphism. Assume that there is a point v in H (3.1) such that $\deg v^*T_X = n$. Then f is an isomorphism.

Proof. By Proposition 3.1.2, the assumption in Proposition 3.4 is satisfied, namely $f^*K_X . L = -n$ for any line L on Q. Hence from the equality: $K_Q = f^*K_X + R_Q$ it follows that $R_Q . L = 0$, and therefore, R_Q is empty because $\mathrm{Pic}\, Q \simeq \mathbb{Z}$. Thus f is umramified, and therefore etale. Proposition 1.5 asserts this theorem.

q.e.d.

In the sequel we assume that

(3.6) $\deg v^*T_X = n+1$. (Proposition 3.1)

Now let $H_0 = \{v \in H \mid v(P^1) \not\subset R_X\}$. Then we see easily that H_0 is an open subset in H. Moreover let $H(=) = \{v \in H_0 \mid v$ is of

$\mp$-type} for $\mp = \alpha, \beta, \gamma$.

Now letting $\mathcal{H}$ ($\subset H \times X \times P^1$) the universal scheme corresponding to the Hilbert scheme H where $p: \mathcal{H} \longrightarrow H$ is the first projection $q: \mathcal{H} \longrightarrow X$ the second projection and $r: \mathcal{H} \longrightarrow P^1$ the third projection. Then we note that $q^* T_X |_{p^{-1}(v)} = v^* T_X$ for each v in H.

Hence we have

Proposition 3.7. Under the above notations, $H(\alpha)$is an open subset in H_0 and $H(\gamma)$ is an open subset in $H_0 - H(\alpha)$.
Proof. Since $p: \mathcal{H} \longrightarrow H$ is flat, we get the former by the semi-continuity of the coherent sheaf. Similarly by considering the coherent sheaf: $q^* T_X \otimes r^* \mathcal{O}(-3)$ we get the latter.

q.e.d.

(3.8) Hence, first we consider the following case:
there is a point P outsides the branched locus R_X and an irreducible component $H_{P,0}$ in H_P such that each point v in $H_{P,0}$ is α-type and the image $v(P^1)$ of P^1 via some element v in $H_{P,0}$ is smooth at the point P.

In this case X is a Fano variety by Proposition 1.3. Thus the proof of Hartshorne's conjecture in §.3 [Mo] shows that X is a projective space.

Next Corollary 3.4.2 and Proposition 3.7 states that the case below does not occur

(3.9) There is a point u in $X - R_X$ such that every element v in some component of the smooth projective scheme $H_u((2)$ of 1.9.1) is of β or γ-type.

Therefore the following case is left to us:

(3.10) For every point u in $X - R_X$, every component of H_u contains an element of α-type and an element of β or γ-type.

Hereafter until the end of this paper we treat with this case which does not occur.

For the purpose we make several preliminaries.

(3.11) Let a point P and an irreducible component $H_{P,0}$ be as in 3.10 and let us define a morphism $\Phi : H_{P,0} \longrightarrow V(T_{X,P}{}^*)$ induced by the canonical morphism : $H \times P^1 \longrightarrow X$

$$H_{P,0} \ni v \longrightarrow dv_{*,o}(\frac{d}{dt}) \in V(T_{X,P}{}^*)$$

where t is a local parameter of P^1 at the fixed point o.

First each element v in $H_{P,0} \cap H(\alpha)$ yields the umramified morphism v: $P^1 \longrightarrow X$. Next, an element v in $H_{P,0} - H(\alpha)$ is induced by the image of some line in Q and

$f: Q \longrightarrow X$ is etale at the points at $f^{-1}(P)$.

Hence we see that Φ is defined as a morphism and $\Phi(H_{P,0}) \subset V(T_{X,P}{}^*) - \{0\}$. Thus we get morphism

$H_{P,0} \longrightarrow P(T_{X,P}{}^*)$ $(\simeq P^{n-1})$ which is G-invariant with $G = \{g \in \mathrm{Aut}\ P^1 \mid g(o) = o\}$ (see 1.14), which induces a canonical surjective morphism θ: $Y \longrightarrow P^{n-1}$ where Y is the geometric quotien

We study a property of the morphism θ.
For an element v in $H_{P,0}$ set the closed subset $\Phi^{-1}(\Phi(v))$ as $F(v)$. Then we have

Proposition 3.12. For an element v in $H_{P,0}$, let w be an element in $F(v)$.

1) if w is of α-type. Then $F(v)$ is smooth at the point w and it is of one-dimension at w.

2) if w is of β or γ type, then $F(v)$ is of one-dimensional at the point w.

Consequently the morphism Φ is equi-dimensional and hence flat.

Proof. 1) is proved in [Mo].

Let us consider (2). Assume that $\dim_w F(v) > 1$. Then by 1) there is a component V ($\ni w$) of $F(v)$ such that every element of V is of β or γ type, which gives the infinite set of rational curves $\{v(P^1) | v \in V\}$ by the fact that $\Gamma : H_{P,0} \longrightarrow Y$ is equi-one-dimensional by Proposition 1.14.

Moreover setting $f^{-1}(P) = \{q_1, \ldots, q_r\}$ we can find a point q_i and an infinite set of lines in Q : $\{$line L in $Q |$ L is an irreducible component of $f^{-1}(v(P^1))$ for some point v in V and passes through the point $q_i\}$ by Corollary 3.2.

Now we remark

(#.) for each i $(1 \leq i \leq r)$, there is a natural isomorphism: ($\ddagger$) $df_{*,q_i}: T_{Q,\, q_i} \simeq T_{X,P}$ and $C(q)$ in Remark 3.4.1 is canonically contained in $P(T_{X,P}^*)$ as a smooth quadric hypersurface.

Therefore we get a contradiction, which yields (2).

q.e.d.

Now let us consider the property of the morphism $\theta: Y \longrightarrow P(T_{X,P}^*)$.

Corollary 3.13. Under the condition and the notations in 3.11, the morphism $\theta: Y \longrightarrow P(T_{X,P}^*)$ is a finite surjective morphism. Moreover $\Gamma(H_{P,0} \cap (H(\beta) \cup H(\gamma))$ (= J) in Y is of dimension $\leq n-2$ and θ is etale at each point in Y - J.

Proof. The first part is obvious by the above proposition 3.11. Hence since f is finite and surjective, the second is obtained by $\ddagger$ in the proof of Proposition 3.12. The last part is proved in [Mo].

q.e.d.

(3.14) Now we divide the case 3.10 into two cases:

(I) There is a point P ($\notin R_X$) such that the closed subset $(H(\beta) \cup H(\gamma)) \cap H_{P,0}$ is of codimension ≥ 2 in $H_{P,0}$. (We have an argument till 3.18)

(II) For every point P ($\notin R_X$) and every component $H_{P,j}$ of H_P, $(H(\beta) \cup H(\gamma)) \cap H_{P,j}$ is of codim 1 in $H_{P,j}$. (This case is argued from 3.19)

We get contradictions to each case in 3.18 and 3.30.

Moreover we have

Corollary 3.15. Assume that the closed subset $(H(\beta) \cup H(\gamma)) \cap H_{P,0}$ is of codimension ≥ 2 in $H_{P,0}$. (see Proposition 3.7). Then the morphism $\theta: Y \longrightarrow P(T^*_{X,P})$ is an isomorphism.

Proof. By Corollary 3.13 the branched part R_θ is empty by the purity of the branched locus. Since P^{n-1} is simply connected, we get the desired fact.

q.e.d.

(3.16) Now the following is studied before the claim 8.2. in [Mo].

Under the above notations, we have a G-invariant morphism:

$F: H_{P,0} \times P^1 \longrightarrow Y \times X$, $F(v,x) = (\Gamma(v), v(x))$, $v \in H_{P,0}$, $x \in P^1$.

Let $Z = \mathrm{Spec}_{Y \times X}[(F_* \mathcal{O}_{H_{P,0} \times P^1})^G]$. Since $Y \simeq P^{n-1}$ and $\Gamma: H_{P,0} \longrightarrow Y$ is equidimensioanl, Γ is flat and therefore universally geometric quotient. Hence

Z is the geometric

quotient $H_{P,0} \times P^1 / G$ and is a P^1 -bundle $\varphi : Z \longrightarrow Y$ for Zariski topology because of a section S of Z over Y , via the morphism $H_{P,0} \longrightarrow (v,P) \in H_{P,0} \times P^1$. Thus we can introduce a proper morphism via the G-invariant morphism π: $H_{P,0} \times P^1$ $(v,x) \longrightarrow v(x) \in X$,

We study the branch locus of the morphism $\pi: Z \longrightarrow X$.

First the following is shown in [Mo].

Remark 3.17. $\pi: Z \longrightarrow X$ is etale at each point of $Z - S - \pi^{-1}((H(\beta) \cup H(\gamma)) \cap H_{P,0})$.

At last we have come to the final stage of the case (I) in 3.14

Now we assume

(3.18) the closed subset $(H(\beta) \cup H(\gamma)) \cap H_{P,0}$ is of codimension ≥ 2 in $H_{P,0}$.

Then by the above remark, we see that π is etale at each point of $Z - S$. Moreover $\pi(S) = P$. Thus, the argument after 8.2 in [Mo] says that $\pi^{-1}(Z - S)$ is finite and etale over $X - \{P\}$ and hence $\pi: Z - S \longrightarrow X - \{P\}$ is an isomorphism and consequently $X \simeq P^n$. Therefore we infer that T_X is an ample vector bundle, which induces a contradiction to the condition I.

Next, we study case II in 3.14.

(3.19) For every point P ($\notin R_X$) and every component $H_{P,j}$ of H_P, $(H(\beta) \cup H(\gamma)) \cap H_{P,j}$ is of codim 1 in $H_{P,j}$.

By Proposition 3.4 and the equality: $K_Q = f^{*}K_X + R_Q$, we see that $R_Q \cdot L = 1$ for any line L on Q. Thus we have an important

Proposition 3.20. The branched locus R_Q is a hyperplane section of Q, namely, it is a smooth one or a cone with an isolated singularity.

Moreover we have

Proposition 3.21. Assume the condition 3.19. Then for every line L on Q, $\deg f^{*}T_{X|L} = n + 1$ and for every line L ($\not\subset R_Q$) on Q, the restriction of $f^{*}T_X$ on the line L is of β or γ -type. Moreover there is no line L on Q so that $f^{*}T_{X|L}$ is ample.

Proof. Proposition 3.4 gives rise to the former.

Next, for every line L ($\not\subset R_Q$) on Q, $f^{*}T_{X|L}$ is GS by Proposition 1. 7. Hence, Proposition 1.2 and the former yield the fact that

($\ddagger$) for every line L ($\not\subset R_Q$) on Q, $f^{*}T_{X|L}$ is of α,β or γ-type.

Now set T as

$\{$a line $L \subset Q |\ f^{*}T_{X|L}$ is of β or γ type$\}$.

Then we claim that

T is a dense subset in the set $\{L \subset Q |\ f(L) \not\subset R_X\}$

Proof. For each point P in $X - R_X$, we can take a point q in Q with $f(q) = P$ and a divisor C_P in $H_{P,j}$ induced by the scmheme $C(q)$ (Remark 3.4.1) by the assumption 3.19 and Corollary 3.4.2.

Then noting that $C(q)$ is contained in T, we get the claim.

In view of the above ($\ddagger$), the type α in the set T is an open condition from Proposition 3.7 . Thus the claim yield the desired fact.

q.e.d.

In the next place, we state a

Remark 3.22. Let V be a hyperplane section in a smooth quadric hypersurface Q in P^{n+1} $(n \geq 3)$. Then for each point P in the regular part of V there is a line passing through the point P and not in V.

Hereafter till the end of this paper let us study the exact sequence of the coherent sheaf on Q:

(3.23) $\quad 0 \longrightarrow T_Q \longrightarrow f^*T_X \longrightarrow M$ (the quotient sheaf) $\longrightarrow 0$

induced by the morphism $f\colon Q \longrightarrow X$.

Note that the support of the coherent sheaf M is R_Q.

Moreover we have

Proposition 3.23. $M \otimes k(x) \simeq k(x)$ for every smooth point x in R_Q . Consequently M is a line bundle on the smooth part of R_Q.

Proof. Restricting the exact sequence 3.23 to a line $L \subset R_Q$, we have the sequence: $\quad T_{Q|L} \xrightarrow{j_L} f^*T_{X|L} \longrightarrow M_{|L} \longrightarrow 0.$

Here $f^*T_{X|L}$ is of β or γ-type. Since the homomorphism j_L is a generical isomorphism, we have the desired fact by Remark 3.22.

q.e.d.

As the matter of fact, we can check that M is a line bundle on R_Q. For the purpose we show

Proposition 3.24. Let U be an open subscheme $(= A^n - \{$one point $= P\})$ and $j\colon U \dashrightarrow A^n$ a natural open immersion. Then, if $n \geq 3$, $R^1 j_* \mathcal{O}_U = 0$. Thus, for an open immersion

$k: Q^n - \{\text{one point}\} \ (= V) \longrightarrow Q^n \ (n \geq 3)$, $R^1 k_* \mathcal{O}_V = 0$.

Proof. Since $R^1 j_* \mathcal{O}_U$ is a quasi-coherent sheaf induced by the module $H^1(U, \mathcal{O}_U)$, it suffices to show that $H^1(U, \mathcal{O}_U) = 0$.
By the theory of the local cohomology, we get the following exact sequence:

$$\longrightarrow H^1_P(A^n, \mathcal{O}) \longrightarrow H^1(A^n, \mathcal{O}) \ (=0) \longrightarrow H^1(U, \mathcal{O}_U)$$
$$\longrightarrow H^2_P(A^n, \mathcal{O}) \ (\simeq H^2_P(P^n, \mathcal{O}) \text{ by the exision where } A^n \text{ is contained naturally in } P^n)$$

Let I be the sheaf of ideal defining the closed point P and $X_m = \mathcal{O}_{P^n} / I^m$ the closed subscheme of P^n. Then there is an exact sequence:

$$0 \longrightarrow I^m / I^{m+1} \longrightarrow \mathcal{O}_{X_{m+1}} \longrightarrow \mathcal{O}_{X_m} \longrightarrow 0.$$

Since $I^m / I^{m+1} = S^m(I / I^2)$ and the support of I / I^2 is P, $H^{n-2}(X_{m+1}, I^m / I^{m+1}) = H^{n-1}(X_{m+1}, I^m / I^{m+1}) = 0$, which implies that $H^{n-2}(X_{m+1}, \mathcal{O}_{X_{m+1}}) \simeq H^{n-2}(X_m, \mathcal{O}_{X_m})$ for each m. Thus we infer that $H^{n-2}(\hat{P}^n, \mathcal{O}_{\hat{P}^n}) \simeq H^{n-2}(X_1, \mathcal{O}_{X_1}) = 0$ where $\hat{P}^n$ is the formal completion of P^n along the point P, which yields the desired fact by the formal duality.

As for the latter since Q^n is covered with finitely many affine spaces, the latter is obtained.

q.e.d.

The above result immediately gives rise to

Corollary 3.24.1. The coherent sheaf M in 3.23 is a line bundle on R_Q.

Proof. If R_Q is smooth, we have nothing to prove. . Next assume that R_Q has the isolated singularity v. Letting a canonical open immersion k: $R_Q - \{v\}$ (= R^o) $\longrightarrow R_Q$, consider the following exact sequence:

$$0 \longrightarrow k_*k^*T_Q \longrightarrow k_*k^*f^*T_X\,|R_Q \longrightarrow k_*k^*M \longrightarrow R^1k_*k^*T_Q.$$

Then Proposition 3.24 asserts this corollary.

q.e.d.

Thus in view of the above corollary, restricting the exact sequence 3.23 to R_Q, we get an exact sequence:

(3.24.2) $$T_Q\,|R_Q \xrightarrow{j'} f^*T_X\,|R_Q \longrightarrow M \longrightarrow 0.$$

Letting Ker j' = L_1 and Cokernel of j' = E, we obtain

(3.24.2) $$0 \longrightarrow L_1 \longrightarrow T_Q|R_Q \longrightarrow E \longrightarrow 0,$$ and

(3.24.3) $$0 \longrightarrow E \longrightarrow f^*T_X\,|R_Q \longrightarrow M \longrightarrow 0.$$

Hence, by Remark 3.22 we have a

Proposition 3.25. Under the condition 3.19, L_1 in the sequence 3.24.2 is an line bundle and E a vector bundle of rank n-1 on R_Q. Moreover $M = L_1 \otimes \mathcal{O}_{R_Q}(1)$.

To show that $M \simeq \mathcal{O}_{R_Q}(2)$, we make preparations.

Proposition 3.26. Under the above notations, $f^{-1}f(R_Q) = R_Q$.

Proof. First we have a

Claim. For a general curve $\bar{C}$ on R_Q, $M_{|\bar{C}}$ is an ample line bundle.

Proof. Assume that $f^{-1}f(R_Q) = R_Q \cup D \cup \ldots$ with divisors $D, \ldots$ Take a general curve C in $f(R^o)$. Then there are two components C_1 and C_2 of $f^{-1}(C)$ where $C_1 \subset R^o$ and C_2 $(\subset D)$ is not a line, $C_2 \not\subset R_Q$. Since $T_{Q|C_2}$ is ample by Theorem 2.1.1, so is $f^*T_{X|C_2}$ (by Proposition 1.1) and therefore so is $T_{X|C}$. Thus since $f^*T_{X|C_1}$ is ample, its quotient line bundle $M_{|C_1}$ is ample.

To complete the proof of our proposition, we have only to show that (\$) there is a line L on Q so that $f^*T_{X|L}$ is ample, (which yields a contradiction to Proposition 3.21)

Thus we divide into two cases:

I) $\dim Q \geq 4$ or $\dim Q = 3$ and R_Q is singular..

II) $\dim Q = 3$ and R_Q is a smooth quadric surface ($\simeq P^1 \times P^1$).

First consider the case I. Taking account of Remark 2.5.1, we infer that M is ample. Hence, Theorem 2.7 asserts the fact (\$).

Next consider the case II. If M is ample, Theorem 2.7 yields \$.

In the second place assume that M is not ample. Then we see that M is GS. In fact since $T_{Q|C_2}$

is GS, so is $f^* T_{X|C_2}$, hence so is $T_{X|C}$ and $M_{|C}$. Thus by the claim we infer that $M \simeq \mathcal{O}(d,0)$ with a positive integer d where $p_i : R_Q \longrightarrow P^1$ is the i-th projection and $\mathcal{O}(a,b) = p_1^* \mathcal{O}_{P^1}(a) \times p_2^* \mathcal{O}_{P^1}(b)$.

Then we can easily check that $T_{Q|R_Q} \simeq T_{R_Q} \oplus N_{R_Q} / Q \simeq \mathcal{O}(2,0) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(1,1)$

By virtue of Proposition 3.25, we have the following sequence:

$$(\#) \quad 0 \longrightarrow \underset{(= \mathcal{O}(d-1,-1))}{L_1} \longrightarrow \underset{(= \mathcal{O}(2,0) \oplus \mathcal{O}(0,2) \oplus \mathcal{O}(1,1))}{T_{Q|R_Q}} \xrightarrow[h]{} E \longrightarrow 0,$$

$$(\#\#) \quad 0 \longrightarrow E \longrightarrow f^* T_{X\,|R_Q} \longrightarrow M_{|R} \; (= \mathcal{O}(d,0)) \longrightarrow 0$$

Now assume that the restriction homomorphism $\bar{h}$: $\mathcal{O}(1,1) \oplus \mathcal{O}(2,0) \longrightarrow E$ of h to the subbundle $\mathcal{O}(1,1) \oplus \mathcal{O}(2,0)$ is generally surgective. Then the restriction $\mathcal{O}(1) \oplus \mathcal{O}(2) \longrightarrow E_{|F}$ of $\mathcal{O}(1,1) \oplus \mathcal{O}(2,0) \longrightarrow E$ to a general fiber F of p_2 implies that $E_{|F}$ is ample. Thus by (##). we see that $f^* T_{X|F}$ is ample which gives \$.

Next assume that $\bar{h}$ is not generally surjective. This give rise to an exact sequence:

$$0 \longrightarrow \mathcal{O}(d-1,-1) \longrightarrow \mathcal{O}(1,1) \oplus \mathcal{O}(2,0) \longrightarrow \mathcal{O}(4-d,2) \longrightarrow 0.$$

Immediately we have d = 1 or 2. Moreover the above exact sequence gives a contradiction because of the fact $H^1(R_Q, \mathcal{O}(a,b)) = 0$ for $a \leq 0$ and $b \leq 0$. Therefore the case does not happen that M is not ample.

Thus we complete our proof.

q.e.d.

Since $K_Q = f^* K_X + R_Q$, the above proposition yields

Corollary 3.26.1. $f^* \mathcal{O}_X(R_X) = \mathcal{O}_Q(2)$ with $R_X = f(R_Q)$.

Now before proving that $f_{|R^o} : R^o \longrightarrow R_X$ is etale where R^o is the smooth part of R_Q, we consider

Proposition 3.27. R_X is smooth at each point of $f(R^o)$. Moreover the inverse image of the smooth part of R_X is R^o.

Proof. Let A be a point in R^o and $B = f(A)$ which yields a canonical local homomorphism $\bar{f} : \mathcal{O}_{X,B} \longrightarrow \mathcal{O}_{Q,A}$. Let x_n be an element in $\mathcal{O}_{Q,A}$ defining the divisor R (around the point A). Then, by Proposition 3.23, we can take a regular parameter $(y_1, \ldots, y_n)$ at the point B such that n-elements $\bar{f}y_1, \ldots, \bar{f}y_{n-1}, x_n$ generate m_A / m_A^2 (m_A the maximal ideal of $\mathcal{O}_{Q,A}$), and therefore they are regular parameter of $\mathcal{O}_{Q,A}$. Now let $z = z(y_1, \ldots, y_n)$ be the local equation of $f(R^o)$ in X around the point B. Then from the fact that $K_Q = f^* K_X + R_Q$ it follows that $\frac{\partial y_n}{\partial x_n} = a\, x_n$ with a unit a in $\mathcal{O}_{Q,A}$.
Moreover we have (#) $\bar{f}\, z = b\, x_n^2$ in $\mathcal{O}_{Q,A}$ with a unit b. To study the regularity of fR^o at the point B, we differenciate # partially by x_n. Then we get
the left-hand side of $\frac{\partial z}{\partial x_n} = \sum_i \frac{\partial z}{\partial y_i} \frac{\partial y_i}{\partial x_n} = \frac{\partial z}{\partial y_n} \frac{\partial y_n}{\partial x_n} = a\, x_n \frac{\partial z_n}{\partial y_n}$ and the right-hand side is $= \frac{\partial b}{\partial x_n} x_n^2 + 2b\, x_n$.
Thus we have $\frac{\partial z_n}{\partial y_n} = \left(\frac{\partial b}{\partial x_n} x_n + 2b \right) / a$, which is a unit at the point B. This is the desired fact. The latter is trivial.

q.e.d.

At last we have come to the final stage.

Letting $\bar{f}$ the restriction map of f to R_Q, we have

Proposition 3.28. Under the above notations, we have $\bar{f}^* \omega_{R_X} = \omega_{R_Q}$. Consequently $\bar{f} : R^o \longrightarrow R_X$ is etale.

Proof. The former is trivial by the equality: $K_Q = f^*K_X + \mathcal{O}_{R_Q}(2)$ and Corollary 3.26.1, which gives rise to the latter.

q.e.d.

Now letting $j: R_Q \longrightarrow Q$ the closed embedding, the morphism $jf: R^o \longrightarrow X$ is umramified. Thus since we have a canonical exact sequence: $0 \longrightarrow T_{R^o} \longrightarrow T_{Q|R^o} \longrightarrow N_{R^o/Q} \ (= \mathcal{O}_{R^o}(1)) \longrightarrow 0$, with $\mathcal{O}_{R^o}(1) = \mathcal{O}_{P^{n+1}}(1)_{|R^o}$ $(R^o \subset R_Q \subset P^{n+1})$, the exact sequences 3.24.1 , 3.24.2 give rise to

Proposition 3.29. $E_{|R^o} = T_{R^o}$ and $L_1 = \mathcal{O}_{R_Q}(1)$. Consequently M. $\simeq \mathcal{O}_{R_Q}(2)$.

Proof. The former is obvious. The latter is obtained by Proposition 3.25.

q.e.d.

Thus the exact sequence 3.28 gives an important exact sequence

(3.30) $\quad 0 \longrightarrow T_Q \longrightarrow f^*T_X \longrightarrow \mathcal{O}_{R_Q}(2) \longrightarrow 0$

Theorem 2.7 asserts that there is a line L such that $f^*T_{X|L}$

is ample, which contradicts the fact of Corollary 3.21.1.

Hence we can show Main Theorem.

Reference

[Ha] R.Hartshorne, Ample vector bundles. Publ.Math. I.H.E.S.29 (1966), 63-94.

[L] R. Lazarsfeld, Some Applications of the Theory of Positive Vector Bundles, Springer Lecture Notes 1092, 29-61.

[Mo] S.Mori, Projective manifolds with ample tangent bundles, Ann. Math. 110 (1979).

[OSS] C.Okonek, M,Schneider und H.Spindler, Vector bundles on complex projective spaces, Boston 1980.

[R.V] R.Remmert and Van de Ven. Über holomorphe Abbildung projectiv-algebraischer Manningfaltigkeiten auf komplexe Raeme, Math, Ann.142 (1961),453-486.

[Sa] E.Sato, A Variety which contains a P^1-fiber space as an ample divisor. Algebraic Geometry and Commutative algebra in honor of Masayoshi Nagata.

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Logarithmic Transformations on Elliptic Fiber Spaces. II.
- elliptic bundle case -

By Yoshio Fujimoto

§ 0. Introduction.

In [11], Kodaira introduced the notion of *logarithmic transformation* and showed that any elliptic surface possessing multiple singular fibers can be reduced to an elliptic surface free from multiple fibers by means of logarithmic transformations.
Moreover he showed that a non - Kähler elliptic surface and a Kähler one can be changed into each other via logarithmic transformations.

Therefore, it is natural to ask whether a similar surjery exists on an elliptic fiber space. Ueno [13] [14] and the author [5][6] considered generalized logarithmic transformations along a non - singular divisor and constructed strange degenerations of surfaces.

In this paper, we shall study " *generalized logarithmic transformations on an elliptic bundle* ".
Here, by an elliptic bundle $f : Y \longrightarrow X$, we mean that Y is a principal fiber bundle over a complex manifold X (not necessarily compact) whose typical fiber and structure groups are non - singular elliptic curves. For an elliptic bundle $f : Y \longrightarrow X$ over X, we shall define generalized logarithmic transformations along an arbitrary Cartier divisor D on X, which is *not necessarily assumed*

to be reduced, irreducible or effective.

Such an attempt was already done by Calabi and Eckmann [2], when they constructed complex structures on the product of two spheres of any odd dimensions. It is an elliptic bundle over the products of two complex projective spaces and cannot admit as an unramified covering a holomorpic $\mathbb{C}^*$ - bundle, while the corresponding result holds for elliptic surfaces. The main purpose of this paper is to prove the counterpart of Kodaira's theorem for elliptic bundles and give explanations for these facts.

We state our main theorem.

Theorem (A.) Let $f : Y \longrightarrow X$ be an elliptic bundle over a projective manifold X, which satisfies either of the following conditions.

(1) Y is Kähler.

or (2) $h^{2,0}(X) = 0$.

Then Y can be obtained from the trivial elliptic bundle over X by a succession of generalized logarithmic transformations in the sense of definition(1.1.).

To prove the above theorem, we need to characterize elliptic bundles which are Kähler. We show that an elliptic bundle $(X \times E)^{\gamma} \longrightarrow X$ over X (, where $E := \mathbb{C}/G$ is a non - singular elliptic curve and $\gamma \in H^1(X, \mathcal{O}(E))$)is Kähler if and only if the first Chern class $c(\gamma) \in H^2(X, G)$ of γ is of finite order. (See Proposition(2.1.) below.)

However, theorem(A) does not necessarily hold if we drop the assumption that Y is Kähler, as was shown by Moriwaki. (See §3.) Furthermore, we can give the characterization of elliptic bundles which are obtained from the trivial bundle by performing generalized logarithmic transformations. (See proposition(1.9.).)

Finally, let us explain briefly the contents of our paper. In § 1, we shall define generalized logarithmic transformations on an elliptic bundle and we give a characterization of elliptic bundles which are obtained from the trivial bundle by performing generalized logarithmic transformations.
Moreover, we shall give a necessary and sufficient condition for an elliptic bundle to admit as an unramified covering a holomorphic $\mathbb{C}^*$- bundle. In § 2, we shall give a criterion of Kählerity for an elliptic bundle and prove theorem(A). In §3, we shall give an example of an elliptic bundle which cannot be obtained from the trivial bundle by performing logarithmic transformations, which was constructed by Moriwaki.

The author wishes to express his thanks to A.Moriwaki for useful suggestions.

Notation and Convention.

By an elliptic fiber space $g : V \longrightarrow W$, we mean that g is a proper surjective morphism of a complex manifold V to a complex manifold W, where each fiber is connected and the general fibers are non - singular elliptic curves. By an elliptic bundle $f : Y \longrightarrow X$, we mean that Y is a principal fiber bundle over X whose typical fiber and structure groups are non - singular elliptic curves.

For a compact complex manifold X, we use the following notation.

$b_i(X)$: the i - th Betti - number of X.

$\kappa(X)$: the Kodaira dimension of X.

K_X : the canonical bundle of X.

$h^{p,q}(X) = \dim_{\mathbb{C}} H^q(X, \Omega^p_X)$.

$q(X) = \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X)$.

$\mathrm{Pic}(X) := H^1(X, \mathcal{O}^*_X)$: the Picard group of X, which has the natural structure of a commutative complex Lie group.

Let $c_1 : H^1(X, \mathcal{O}^*_X) \longrightarrow H^2(X, \mathbb{Z})$ be the first Chern class map.

$\mathrm{Pic}^0(X) := \mathrm{Ker}(c_1)$: the identity component of Pic(X).

X is in the class $\mathscr{C}$ in the sense of Fujiki if X is a meromorphic image of a compact Kähler manifold.

If D is a divisor on X, we set

[D] : the line bundle on X determined by D.

$c_1([D])$: the first Chern class of the line bundle [D].

supp(D) : the support of D.

For a non - singular elliptic curve E, we represent E as a quotient group : $E = \mathbb{C}/G$, where G is a discontinuous subgroup of the additive group $\mathbb{C}$ generated by τ and 1, $\mathrm{Im}(\tau) > 0$, and for any $\zeta \in \mathbb{C}$, we denote by $[\zeta]$ the corresponding element of $E = \mathbb{C}/G$.

$\mathcal{O}_X(E)$: the sheaf over X of germs of holomorphic functions with values in E.

§ 1. Generalized Logarithmic Transformations on an Elliptic Bundle.

Let $f : Y \longrightarrow X$ be an elliptic bundle over a complex manifold X (not necessarily compact) whose typical fiber and structure group is the elliptic curve $E = \mathbb{C} / G$, $G := \mathbb{Z} \oplus \mathbb{Z}\tau$, $\mathrm{Im}(\tau) > 0$.
Then Y can be expressed in the form : $Y = (X \times E)^{\eta}$ for some $\eta \in H^1(X, \mathcal{O}(E))$. (Here we follow the notation of Kodaira[11].) And let D be an arbitrary Cartier divisor on X and take $\alpha \in G$ arbitrarily. Choose a sufficiently fine open covering $\{U_i\}_{i \in I}$ of X such that $s_i = 0$ is the local defining equation of D on U_i.
The transition function of [D] is expressed by a cocycle $\{f_{ij}\}$ $(:= s_i/s_j)$ and $\eta \in H^1(X, \mathcal{O}(E))$ is expressed by a cocycle $\{\eta_{ij}\}$ with respect to the covering $\{U_i\}$ of X.

Now, identify $(z, [\eta_i]) \in U_i \times E$ with $(z, [\eta_j]) \in U_j \times E$ if and only if $[\eta_i] = [\, \eta_j + \dfrac{\alpha}{2\pi\sqrt{-1}} \log(f_{ij}) + \eta_{ij} \,]$.
By patching $U_i \times E$'s in this way, we obtain a new elliptic bundle Z over X. Outside the support of D, there exists an isomorphism

$$\begin{array}{cccc} \Lambda : & Z|_{X \setminus \mathrm{supp}(D)} & \xrightarrow{\sim} & Y|_{X \setminus \mathrm{supp}(D)} \\ & \Psi & & \Psi \\ & (z, [\eta_i]) & \longrightarrow & (z, [\, \eta_i - \dfrac{\alpha}{2\pi\sqrt{-1}} \log(s_i)] \,) . \end{array}$$

In fact, on $U_i \cap U_j$ we have :

$$[\, \eta_i - \frac{\alpha}{2\pi\sqrt{-1}} \log(s_i) \,] = [\, \eta_j - \frac{\alpha}{2\pi\sqrt{-1}} \log(s_j) + \eta_{ij} \,] .$$

And the elliptic bundle over each irreducible component of D is determined by the restriction of the line bundle [D].
Hence Z can be obtained from Y by performing surjeries along D.

Definition (1.1.) We write $Z = L_D(\alpha)(Y)$ and call $L_D(\alpha)$ *a generalized logarithmic transformation along D.*

Remark(1.2.) In the above definition, D is not necessarily assumed to be reduced, irreducible or effective. However by changing $\alpha \in G$ suitably and performing successive logarithmic transformations, Z can also be written in the form:

$Z = L_{D_1}(\alpha_1) L_{D_2}(\alpha_2) \cdots L_{D_k}(\alpha_k)(Y)$, where $\alpha_i \in G$ and each D_i is an irreducible, reduced and effective divisor on X.

Remark (1.3.) Let H be a non - singular analytic arc in X which intersects with D transversally. Then when restricted to the elliptic surface over H, the above surjery just coincides with the original logarithmic transformations defined by Kodaira [11].

Remark (1.4.) By using generalized logarithmic transformations, Ueno[14] constructed non - Kähler degenerations of complex tori. Other examples of strange degenerations of surfaces can be found in Ueno[13] and Fujimoto[5].

Remark(1.5.) Calabi and Eckmann [2] constructed a complex structure M on $S^{2m+1} \times S^{2n+1}$ with $m \geq 1$ and $n \geq 1$, which is the simplest example of simply connected, compact homogeneous non - Kähler manifolds. It is an elliptic bundle over $\mathbb{P}^m \times \mathbb{P}^n$ and can be written in the form : $M = L_{H_1}(-1)L_{H_2}(-\tau)\ (\mathbb{P}^m \times \mathbb{P}^n \times E)$,

$H_1 = p_1^* \mathcal{O}_{\mathbb{P}^m}(1)$, $H_2 = p_2^* \mathcal{O}_{\mathbb{P}^n}(1)$, where $p_1 : \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^m$ (resp. $p_2 : \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^n$) is the projection to the first (resp. second) factor.

In order to prove theorem(A), we need the following proposition.

Proposition(1.6.) Let $f : Y = (X \times E)^{\eta} \longrightarrow X$ be an elliptic bundle over a projective manifold X with $E \simeq \mathbb{C}/(1,\tau)$, $\mathrm{Im}(\tau) > 0$ and $\eta \in H^1(X,\mathcal{O}(E))$. Assume that the first Chern class $c(\eta) \in H^2(Y,G)$ of η is *of finite order*. Then Y can be obtained from the trivial elliptic bundle over X by a succession of generalized logarithmic transformations.

Proof. From the exact sequence $0 \longrightarrow G \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(E) \longrightarrow 0$, we have the following exact sequence of cohomology:

$$\longrightarrow H^1(X,G) \longrightarrow H^1(X,\mathcal{O}_X) \xrightarrow{\ p\ } H^1(X,\mathcal{O}_X(E)) \xrightarrow{\ c\ } H^2(X,G) \longrightarrow \cdots.$$

$\eta \in H^1(X,\mathcal{O}_X(E))$ is expressed by a cocycle $\{\eta_{ij}\}$ with respect to the open covering $\{U_i\}_{i\in I}$ of X. And Y is obtained by identifying $(z,[\xi_i]) \in U_i \times E$ and $(z,[\xi_j]) \in U_j \times E$ if and only if

$$[\xi_i] = [\xi_j + \eta_{ij}(z)].$$

Step 1. We first consider the case where $c(\eta) = 0$.
Then there exists $\tilde{\eta} = \{\tilde{\eta}_{ij}(z)\} \in H^1(X, \mathcal{O}_X)$ with $p(\tilde{\eta}) = \eta$.
Since the structure of Y is uniquely determined by the cohomology class of $\eta \in H^1(X,\mathcal{O}(E))$, we may assume that $[\tilde{\eta}_{ij}(z)] = \eta_{ij}(z)$.
By identifying $(z, \xi_i) \in U_i \times \mathbb{C}$ with $(z, \xi_j) \in U_j \times \mathbb{C}$ if and only if $\xi_i = \xi_j + \tilde{\eta}_{ij}(z)$, we obtain an affine bundle W over X. $\mathbb{Z}^2$ acts on W by

$$\begin{array}{cccc} (k,\ \ell) : & W & \longrightarrow & W, \\ & \cup\!\!\!\!\!\!\;\cup & & \cup\!\!\!\!\!\!\;\cup \\ & (z,\ \xi_i) & \longrightarrow & (z,\ \xi_i + k + \ell\tau) \end{array}$$

and Y is isomorphic to the quotient space $Y/_{\mathbb{Z}^2}$.

Now, we have the following exact sequence:

$$\longrightarrow H^1(X,\mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \xrightarrow{e} H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z}) .$$

If we put $L := e(\tilde{\eta})$, we have $L \in \mathrm{Pic}^\circ(X)$ and the transition function of L can be expressed by a cocycle $\{ \exp(2\pi\sqrt{-1}\, \tilde{\eta}_{ij}(z)) \}$ with respect to the covering $\{U_i\}$ of X.

Let $\mathbb{L}$ be the total space of the line bundle L and define an automorphism g of $\mathbb{L}^* := \mathbb{L} \setminus \{ \text{zero - section} \}$ by

$$\begin{array}{cccc} g : & \mathbb{L}^* & \longrightarrow & \mathbb{L}^* , \\ & \Psi & & \Psi \\ & (z, \eta_i) & \longrightarrow & (z, \rho\eta_i) \end{array}$$

where we put $\rho = \exp(2\pi\sqrt{-1}\ \tau)$ ($|\rho| < 1$) and $\mathbb{L}|_{U_i}$ has local coordinates (z , η_i). The group $\langle g \rangle$ acts on $\mathbb{L}^*$ freely and properly discontinuously, so the quotient space $Z := \mathbb{L}^*/_{\langle g\rangle}$ is non - singular. There is a natural morphism

$$\begin{array}{cccc} \Phi : & Z & \longrightarrow & X , \\ & \Psi & & \Psi \\ & \overline{(z, \eta_i)} & \longrightarrow & z \end{array}$$

where by $\overline{(z, \eta_i)}$, we denote the point of Z corresponding to a point $(z, \eta_i) \in \mathbb{L}^*$. By this morphism, Z is an elliptic bundle over X and we have an isomorphism

$$\begin{array}{cccc} \Lambda : & Y & \xrightarrow{\sim} & Z . \\ & \Psi & & \Psi \\ & (z, [\xi_i]) & \longrightarrow & \overline{(z, \exp(2\pi\sqrt{-1}\xi_i))} \end{array}$$

Since X is projective, there exists a Cartier divisor D on X with $L \simeq \mathcal{O}_X(D)$. Hence by the above isomorphism Λ and definition(1.1.), we have $Y \simeq L_D(1)(X \times E)$, that is, Y can be obtained from $X \times E$ by means of generalized logarithmic transformations along D.

Step 2. Next, we consider the general case.

There is a natural isomorphism $G \simeq \mathbb{Z} \oplus \mathbb{Z}$ and $H^2(X,G) \simeq H^2(X,\mathbb{Z}) \oplus$

$H^2(X,\mathbb{Z})$. The first Chern class $c(\eta) \in H^2(X,G)$ of η can be expressed by a cocycle $\{c_{ijk}\}$, $c_{ijk} = p_{ijk} + \tau\cdot q_{ijk}$, p_{ijk}, $q_{ijk} \in \mathbb{Z}$, with respect to the covering $\{U_i\}$ of X.
Then $c' := \{p_{ijk}\} \in H^2(X,\mathbb{Z})$ and $c'' := \{q_{ijk}\} \in H^2(X,\mathbb{Z})$ are of finite order, since $c(\eta) \in H^2(X,G)$ is of finite order.

Hence there exists Cartier divisors D_1, D_2 on X with $c_1([D_1]) = c'$, $c_1([D_2]) = c''$. By performing generalized logarithmic transformations along D_1 and D_2, we obtain a new elliptic bundle $Y' := L_{D_1}(-1)L_{D_2}(-\tau)(Y)$ over X. If we put $Y' = (X \times E)^{\eta'}$, $\eta' \in H^1(X, \mathcal{O}(E))$, we have $c(\eta') = 0$. Hence from step 1, Y' can also be obtained from a trivial elliptic bundle by successive logarithmic transformations. Since $Y = L_{D_1}(1)L_{D_2}(\tau)(Y')$, the same is clearly true for the original Y. <u>q.e.d.</u>

<u>Remark(1.7.)</u> The assumption that $c(\eta)$ is of finite order means that Y is Kähler. This will be shown in § 2.

<u>Remark(1.8.)</u> In case $c(\eta) = 0$, the elliptic bundle $f : Y \longrightarrow X$ can be defined by locally constant transition functions.

By the same argument as in the proof of proposotion(1.6.), we get the following proposition.

<u>Proposition(1.9.)</u> Let $f: Y = (X \times E)^{\eta} \longrightarrow X$ be an elliptic bundle over a projective manifold X with $E \simeq \mathbb{C}/(1, \tau)$, $\mathrm{Im}(\tau) > 0$ and $\eta \in H^1(X, \mathcal{O}(E))$. Then Y can be obtained from the trivial bundle over X by generalized logarithmic transformations if and only if the following condition is satisfied.

Condition(A). Via the natural isomorphism

$$\begin{array}{ccc} \Lambda : H^2(X,G) & \simeq & H^2(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}), \\ \Psi & & \Psi \\ c(\eta) & \longrightarrow & (c', c'') \end{array}$$

c', $c'' \in H^2(X, \mathbb{Z})$ are both algebraic cycles.

Corollary(1.10.) Let $f: Y \longrightarrow X$ be an elliptic bundle over a projective manifold X with $h^{2,0}(X) = 0$. Then Y can be obtained from the trivial bundle over X by means of logarithmic transformations.

Proof. Since $h^{2,0}(X) = 0$, $H^1(X,\mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z})$ is surjective and the claim follows immediately from proposition(1.9.).

Example(1.11.) The following example is due to M.Kato [8], p.55, example 2. Let $\ell = \{(z_0: z_1: z_2: z_3) \in \mathbb{P}^3 \mid z_0 = z_1 = 0\}$, $\ell_\infty = \{(z_0: z_1: z_1: z_3) \in \mathbb{P}^3 \mid z_2 = z_3 = 0\}$ and put $W := \mathbb{P}^3 \setminus \ell \setminus \ell_\infty$. Fix a constant $\alpha \in \mathbb{C}$ with $0 < |\alpha| < 1$ and define an automorphism g of W by : $g : (z_0: z_1: z_2: z_3) \longrightarrow (z_0: z_1: \alpha z_2: \alpha z_3)$.
Then the quotient space $M := W /_{\langle g \rangle}$ is a compact complex manifold of *class L*. By a natural holomorphic map $p : M \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ sending $\overline{(z_0: z_1: z_2: z_3)}$ to $((z_0: z_1), (z_2: z_3))$, M is an elliptic bundle over $\mathbb{P}^1 \times \mathbb{P}^1$, where by $\overline{(z_0: z_1: z_2: z_3)}$, we denote the point of M corresponding to a point $(z_0: z_1: z_2: z_3) \in W$.
Since $H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(E)) \simeq H^2(\mathbb{P}^1 \times \mathbb{P}^1, G) \simeq G \oplus G$, M can be written in the form : $M = L_{H_1}(-1)\, L_{H_2}(1)(\mathbb{P}^1 \times \mathbb{P}^1 \times E)$, $H_i \simeq p_i^* \mathcal{O}_{\mathbb{P}^1}(1)$, where $p_i : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is the projection to the i - th factor. It is easy to see that an elliptic surface over the curve of type (1,1) on $\mathbb{P}^1 \times \mathbb{P}^1$ is trivial and contains many rational curves.

Proposition (1.12.) Let $Y = L_{D_1}(1)\, L_{D_2}(\tau)(X \times E)$ be an elliptic bundle over a projective manifold X, where $E \simeq \mathbb{C}/(1, \tau)$, $\mathrm{Im}(\tau) > 0$ and D_1, D_2 are Cartier divisors on X. Then Y is isomorphic to the $\mathbb{Z}$ - quotient of some $\mathbb{C}^*$ - bundle over X if and only if there exists a Cartier divisor H on X with $c_1([D_1]) = \alpha \cdot c_1([H])$, $c_1([D_2]) = \beta \cdot c_1([H])$ for some $\alpha, \beta \in \mathbb{Z}$.

Proof. (*necessity*) Assume that Y is isomorphic to the $\mathbb{Z}$ - quotient Y' of some $\mathbb{C}^*$- bundle on X. Then Y' can be written in the form : $Y' = L_D(1)(X \times E')$, where $E' \simeq \mathbb{C}/(1,\tau')$, $\mathrm{Im}(\tau') > 0$ and D is a Cartier divisor on X. Take an open covering $\{U_i\}$ of X such that $Y|_{U_i} \simeq U_i \times E$, $Y'|_{U_i} \simeq U_i \times E'$ and $Y|_{U_i}$ (resp. $Y'|_{U_i}$) has coordinates $(z, [\xi_i])$ (resp. $(z, [\eta_i])$). On $U_i \cap U_j$, we have

(1) $[\ \xi_i\ ] = [\ \xi_j + \dfrac{1}{2\pi\sqrt{-1}} \log(f_{ij}) + \dfrac{\tau}{2\pi\sqrt{-1}} \log(g_{ij})\ ]$

(2) $[\ \eta_i\ ] = [\ \eta_j + \dfrac{1}{2\pi\sqrt{-1}} \log(h_{ij})\ ]$,

where $\{f_{ij}\}$ (resp. $\{g_{ij}\}$, $\{h_{ij}\}$) is the transition function of the line bundle $[D_1]$ (resp. $[D_2]$, $[D]$). Since the bundle structures of Y and Y' are uniquely determined by its cohomology class, we may assume that the above isomorphism over X can be written as follows.

(3) $[\ \xi_i] = [\ \dfrac{\eta_i}{c\tau' + d}\ ]$, $\tau = \dfrac{a\tau' + b}{c\tau' + d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$.

From (2) and (3), we have $[\ \xi_i\ ] = [\ \xi_j + \dfrac{-c\tau + a}{2\pi\sqrt{-1}} \log(h_{ij})]$,

since $\dfrac{1}{c\tau' + d} = -c\tau + a$. In view of (1) and the natural isomorphism $H^2(X, G) \simeq H^2(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})$, we have

$c_1([D_1]) = a \cdot c_1([H])$, $c_1([D_2]) = -c \cdot c_1([H])$, $a,c \in \mathbb{Z}$.

(*sufficiency*) We chosse integers α', β', c, d and k such that $c\alpha' - \beta' d = 1$, $\alpha = k\alpha'$, $\beta = k\beta'$ and introduce a new fiber coordinate $\xi' = \dfrac{\xi}{\alpha' + \beta'\tau}$ on $U_i \times \mathbb{C}$. Correspondingly we replace τ by $\tau' = \dfrac{c\tau + d}{\beta'\tau + \alpha'}$ and put $G' \simeq \mathbb{Z} \oplus \mathbb{Z}\tau'$, $E' \simeq \mathbb{C}/G'$.

Then the formula (1) takes the form

(2) $[\ \xi'_i\ ] = [\ \xi'_j + \eta'_{ij}\ ]$,

where $[\ \eta'_{ij}\ ] = \left[\ \dfrac{\frac{1}{\beta'\tau + \alpha'}}{2\pi\sqrt{-1}} \log(f_{ij}) + \dfrac{\frac{\tau}{\beta'\tau + \alpha'}}{2\pi\sqrt{-1}} \log(g_{ij})\ \right]$,

$\eta' = \{\ \eta'_{ij}\ \} \in H^1(X, \mathcal{O}(E'))$.

Hence Y is isomorphic to the elliptic bundle $(\ X \times E')^{\eta'}$ and the first Chern class $c(\eta') \in H^2(X, G')$ of η' is as follows.

$$\begin{aligned} c(\eta') &= \frac{1}{\alpha' + \beta'\tau} c_1([D_1]) + \frac{\tau}{\alpha' + \beta'\tau} c_1([D_2]) \\ &= \frac{\alpha + \beta\tau}{\alpha' + \beta'\tau} c_1([H]) \\ &= k \cdot c_1([H]), \quad k \in \mathbb{Z}. \end{aligned}$$

Therefore, by the proof of Proposition (1.6.), step 1, there exists a Cartier divisor D on X such that $Y \simeq L_H(k) L_D(1)(X \times E')$, $k \in \mathbb{Z}$. Hence if we put $L = [\ kH + D\ ]$, Y is isomorphic to the $\mathbb{Z}$ - quotient of the $\mathbb{C}^*$- bundle associated to L. q.e.d.

Corollary (1.13.) Every elliptic bundle over a curve admits as its unramified covering a holomorphic $\mathbb{C}^*$- bundle, while the correspondig result does not hold for Calabi - Eckmann manifolds. (c.f. Remark(1.5.))

Proposition(1.14.) Let X be a complex manifold (which are not necessarily compact) and $X^\circ \subset X$ be a Zariski open subset with $\operatorname{codim}(X \setminus X^\circ) \geq 3$. Suppose that there exists an elliptic bundle $f^\circ : Y^\circ \longrightarrow X^\circ$ over X°. Then there exists a relative compactication $f : Y \longrightarrow X$.

<u>Proof.</u> From the exact sequence

$$0 \longrightarrow G \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(E) \longrightarrow 0,$$

we have the following commutative diagram.

$$\begin{array}{ccccccccc}
\longrightarrow & H^1(X, \mathcal{O}_X) & \longrightarrow & H^1(X, \mathcal{O}_X(E)) & \longrightarrow & H^2(X,G) & \longrightarrow & H^2(X,\mathcal{O}_X) \\
& \downarrow & & \downarrow & & \| & & \downarrow \\
\longrightarrow & H^1(X^\circ, \mathcal{O}_{X^\circ}) & \longrightarrow & H^1(X^\circ, \mathcal{O}_{X^\circ}(E)) & \longrightarrow & H^2(X^\circ, G) & \longrightarrow & H^2(X^\circ,\mathcal{O}_{X^\circ})
\end{array}$$

Since $\operatorname{codim}(X \setminus X^\circ) \geq 3$, by the theorem of Siu - Trautmann, we have an isomorphism $H^1(X, \mathcal{O}_X) \simeq H^1(X^\circ, \mathcal{O}_{X^\circ})$ and the restriction map $H^2(X, \mathcal{O}_X) \longrightarrow H^2(X^\circ, \mathcal{O}_{X^\circ})$ is injective. Hence from the above diagram, the restriction map $r : H^1(X, \mathcal{O}_X(E)) \longrightarrow H^1(X^\circ, \mathcal{O}_{X^\circ}(E))$ is surjective. If we put $Y^\circ = (X^\circ \times E)^{\gamma^\circ}$, $\gamma^\circ \in H^1(X^\circ, \mathcal{O}_{X^\circ}(E))$, there exists $\gamma \in H^1(Y, \mathcal{O}_Y(E))$ with $r(\gamma) = \gamma^\circ$.
Then $X := (X \times E)^\gamma \longrightarrow X$ is a relative compactification of $f^\circ : Y^\circ \longrightarrow X^\circ$. <u>q.e.d.</u>

The above proposition is not necessarily true if $\operatorname{codim}(Y \setminus Y^\circ) = 2$. The following example is due to K.Ueno.

<u>Example (1.15.)</u> Let $X = \mathbb{C}^2 \setminus 0$ and $U_0 = \{(x,y) \in \mathbb{C}^2 \mid x \neq 0\}$, $U_1 = \{(x,y) \in \mathbb{C}^2 \mid y \neq 0\}$ be open coverings of X.
By patching $(x,[\xi]) \in U_0 \times E$ and $(y,[\xi']) \in U_1 \times E$ if and only if $[\xi'] = [\xi + \frac{1}{xy}]$, we obtain an elliptic bundle Y over X.

Since $\{\frac{1}{xy}\} \in H^1(X, \mathcal{O}) \simeq H^1(X, \mathcal{O}(E))$ and $H^1(\mathbb{C}^2, \mathcal{O}(E)) = 0$, the restriction map $H^1(\mathbb{C}^2, \mathcal{O}(E)) \longrightarrow H^1(X, \mathcal{O}(E))$ is not surjective. It is not known whether it has a relative compactification or a Kähler metric.

§ 2. A Criterion of Kählerity for an elliptic bundle and a proof of Main Theorem(A).

In this section, we shall give a necessary and sufficient condition for an elliptic bundle to be Kähler and prove main theorem(A). The following theorem is fundamental.

Theorem (2.1.) (Fujiki [3], Proposition(4.3.), Lemma(4.4.))
Let $f : X \longrightarrow Y$ be a fiber space of compact complex manifolds in the class $\mathscr{C}$. Let $U \subset Y$ be a Zariski open subset over which f is smooth and $f_U : X_U \longrightarrow U$ be the restriction of f to X_U.
If $R^1 f_{U_*} \mathbb{C}$ is a constant system, then $q(X) = q(Y) + q(f)$, where $q(f)$ denotes the irregularity of the general fiber of f.

Proposition(2.2.) Let $f : Y := (X \times E)^{\eta} \longrightarrow X$ be an elliptic bundle over a projective manifold X with $E \simeq \mathbb{C}/G$, $G \simeq \mathbb{Z} \oplus \mathbb{Z}\tau$, $\mathrm{Im}(\tau) > 0$ and $\eta \in H^1(X, \mathcal{O}(E))$.
Then the following conditions are equivalent.

(1) Y is Kähler.

(2) Y is in the class $\mathscr{C}$.

(3) $b_1(Y) = b_1(X) + 2$.

(4) The first Chern class $c(\eta) \in H^2(X, G)$ of η is of finite order m in $H^2(X, G)$.

Moreover, if $q(X) = 0$, (1) is also equivalent to the following condition.

(5) Y is projective.

<u>Proof.</u> (1) $\Longrightarrow$ (2). Obvious.

(2) $\Longrightarrow$ (3). This follows immediately from theorem(2.1.).

(3) $\Longrightarrow$ (4). Assume the contrary.

Then we have $\alpha(c(\eta)) \neq 0$, where $\alpha : H^2(X,G) \longrightarrow H^2(X,\mathbb{C})$ is the natural homomorphism. Hence by the same method as in the proof of [11], III, theorem(11.9.), we have $b_1(Y) \leq b_1(X) + 1$. This is a contradiction.

(4) $\Longrightarrow$ (1). We follow the idea of Miyaoka [12].

Since the Kählerity is preserved under a finite, ètale morphism, it suffices to prove that $(X \times E)^{m\eta}$ is Kähler. Replacing η by $m\eta$, we may assume that $c(\eta) = 0$. Hence from the exact sequence

$$(*)\ H^1(X, G) \xrightarrow{i^*} H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X(E)) \xrightarrow{c} H^2(X, G) \longrightarrow H^2(X,\mathcal{O}_X)\ ,$$

we have $\eta \in H^1(X, \mathcal{O}_X)/_{i^*}H^1(X, G)$.

Since X is projective, we have

$j^*H^1(X, \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \mathbb{R} \simeq H^1(X, \mathcal{O}_X)$, where $j^* : H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X)$ is the natural homomorphism. Moreover, from the natural isomorphism $H^1(X, G) \simeq H^1(X, \mathbb{Z}) \oplus H^1(X, \mathbb{Z})$, it follows that

$i^*H^1(X, G) \underset{\mathbb{Z}}{\otimes} \mathbb{R} \simeq H^1(X, \mathcal{O}_X)$. Then from the sequence $\{k\eta\}_{k = 1,2,\cdots}$ we can take a subsequence $\{\eta_k\}_{k = 1,2,\cdots}$, which converges to η_0

with a finite order. Hence for a suitable integer m, $B^{m\eta}$ is a small deformation of B^{η_0}, which is projective.

This implies that $B^{m\eta}$ and hence B^{η} are Kähler.

(1) $\Longrightarrow$ (5). Moreover, if $q(X) = 0$, then the homomorphism $c : H^1(X, \mathcal{O}(E)) \longrightarrow H^2(X, G)$ is injective and (4) implies that $\eta \in H^1(X, \mathcal{O}(E))$ is of finite order. Hence Y is projective.

q.e.d.

Remark(2.3.) In general, $H^2(X, G)$ has torsions, so the homomorphism $\alpha : H^2(X, G) \longrightarrow H^2(X, \mathbb{C})$ is not necessarily injective.

The following corollary follows immediately from Proposition(2.2.).

Corollary(2.4.) Let $f : Y = L_{D_1}(\alpha_1)\cdots L_{D_k}(\alpha_k)(X \times E) \longrightarrow X$ be an elliptic bundle over a projective manifold X, where $\alpha_i = m_i + n_i\tau \in G$, and $D_1, \cdots, D_k$ are Cartier divisors on X, (which are not necessarily reduced, irreducible or effective.) Then Y is Kähler if and only if (*) $c_1([\sum_{i=1}^{k} m_i D_i])$, $c_1([\sum_{i=1}^{k} n_i D_i]) \in H^2(X, \mathbb{Z})$ are of finite order.

Remark(2.5.) If $q(X) > 0$, (*) does not necessarily imply that Y is projective. There exists a two dimensional complex torus of algebraic dimension one. An algebraic reduction gives it the structure of an elliptic bundle over an elliptic curve, which can be defined by locally constant transition functions.

Corollary(2.6.) (c.f. Katsura and Ueno [9], Appendix 1)

The elliptic surface $f : S = L_{p_1}(m_1, \alpha_1)\cdots L_{p_\lambda}(m_\lambda, \alpha_\lambda)(C \times E) \longrightarrow C$ over a compact curve C is Kähler if and only if $\sum_{i=1}^{\lambda} \alpha_i = 0$.

Moreover if $C \simeq \mathbb{P}^1$, this implies that S is projective.

Proof. Let m be the least common multiple of m_i's$(1 \le i \le \lambda)$. The multiplication map $m : E \longrightarrow E$ induces a finite surjective morphism $\Phi : S \longrightarrow Y$, where $Y = L_{p_1}(1,m\alpha_1)\cdots L_{p_\lambda}(1,m\alpha_\lambda)(C \times E)$ is an elliptic bundle. Hence S is Kähler if and only if Y is Kähler. Corollary(2.4.) implies that Y is Kähler if and only if $\sum_{i=1}^{\lambda} \alpha_i = 0$. Hence the claim follows. q.e.d.

Proposition(2.7.) Let $f : Y \longrightarrow X$ be an elliptic bundle over a projective manifold X with $\dim(X) \ge 3$. Assume that for any p - dimensional submanifold Z on X (, where $p \ge 2$ is a fixed positive integer), the elliptic bundle $Y|_Z \longrightarrow Z$ over Z is Kähler.

Then Y is also Kähler.

Proof. There exists $\eta \in H^1(X, \mathcal{O}(E))$ with $Y \simeq (X \times E)^{\eta}$, $E \simeq \mathbb{C}/G$, $G \simeq \mathbb{Z} \oplus \mathbb{Z}\tau$, $\mathrm{Im}(\tau) > 0$. Via the natural isomorphism $H^2(X, G) \simeq H^2(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})$, the first Chern class $c(\eta) \in H^2(X, G)$ of η can be expressed as $c(\eta) = (c', c'')$, where $c', c'' \in H^2(X, \mathbb{Z})$. Let V be a smooth hyperplane section of X. Then by Lefshetz's hyperplane section theorem, the map $H^2(X, \mathbb{Q}) \longrightarrow H^2(V, \mathbb{Q})$ induced by the inclusion $\iota : V \hookrightarrow X$ is an isomorphism if $\dim(X) \ge 4$ and injective if $\dim(X) = 3$. Continuing this process successively, we find a non - singular surface $S \subset X$ such that the natural homomorphism $H^2(X, \mathbb{Q}) \longrightarrow H^2(S, \mathbb{Q})$ is injective. Since $Y|_S$ is Kähler, proposition(2.2.) implies that $c'|_S, c''|_S \in H^2(S, \mathbb{Z})$ are of finite order. Hence $c', c'' \in H^2(X, \mathbb{Z})$ are of finite order. Therefore, by proposition(2.2.), Y is Kähler. q.e.d

Remark(2.8.) If we drop the assumption that $p \ge 2$, the above proposition does not necessarily hold. (See Remark(3.3.).)

Proof of theorem(A). This follows immediately from proposition (1.6.), corollary (1.10.) and proposition (2.2.).

Corollary (B.) Under the same assumptions as in theorem(A), there exists a Zariski open subset U of X such that the restriction $Y|_U \longrightarrow U$ over U is a trivial elliptic bundle.

Example(2.9.) Let $f : Y = (X \times E)^{\eta} \longrightarrow X$ be an elliptic bundle over an n - dimensional projective manifold X. The first Chern class $c(\eta) \in H^2(X, G)$ can be expressed as $c(\eta) = (c', c'') \in H^2(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})$. Put $N_{\mathbb{Q}}(X) = (\{\text{one - cycles on } X\}/ \equiv) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\equiv$ denotes the numerically equivalence class. Via the intersection pairing (.) of one - cycles and Cartier divisors, $N_{\mathbb{Q}}(X)$ is dual to $NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, where NS(X) denotes the Neron - Severi group of X. Then via the cup - product $\langle \ , \ \rangle : H^2(X, \mathbb{Z}) \otimes H^{2n-2}(X, \mathbb{Z}) \longrightarrow \mathbb{Z}$, $-c'$ (resp. $-c''$) is a linear functional on $N_{\mathbb{Q}}(X)$ and can be identified with an element D'(resp. D'') of $NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Namely,

(**) $\langle x, -c' \rangle = (x , D')$ (resp. $\langle x, -c'' \rangle = (x, D'')$) for
for all $x \in N_{\mathbb{Q}}(X)$.)

Choose a positive integer r such that $D_1 = rD'$ and $D_2 = rD''$ are integral divisors and put $W := L_{D_1}(1)L_{D_2}(\tau)(X \times E)^{r\eta}$.

Then by proposition (2.2.), the elliptic bundle $W|_C \longrightarrow C$ is kähler for all non - singular curve $C \subset X$.

§ 3. Counterexamples.

In this section, we shall construct an elliptic bundle $f : Y \longrightarrow X$ over a projective manifold X which cannot be obtained from the trivial bundle over X by generalized logarithmic transformations.

We first need the following lemma.

Let $E \simeq \mathbb{C}/(1, \tau)$, $\mathrm{Im}(\tau) \neq 0$ be a non - singular elliptic curve and put $G := \mathbb{Z} \oplus \mathbb{Z}\tau$. There is the following exact sequence :

$$\longrightarrow H^1(X, \mathcal{O}(E)) \xrightarrow{c} H^2(X, G) \xrightarrow{q} H^2(X, \mathcal{O}_X) \longrightarrow$$

Lemma(3.1.) For $\alpha, \beta \in H^2(X, \mathbb{Z})$, we have $\alpha + \beta\tau \in \mathrm{Im}(c)$ if and only if (*) $\alpha_{(2,0)} + \beta_{(2,0)}\bar{\tau} = 0$, where $\alpha_{(2,0)}$ (resp. $\beta_{(2,0)}$) means the (2, 0) - component of α (resp. β) under the Hodge decomposition of $H^2(X, \mathbb{C})$.

Proof. We have the following commutative diagram.

$$\begin{tikzcd} \longrightarrow H^1(X, \mathcal{O}(E)) \arrow[r, "c"] & H^2(X, G) \arrow[r, "p"] \arrow[d, "j"] & H^2(X, \mathcal{O}_X) \simeq H^{0,2} \\ & H^2(X, \mathbb{C}) \simeq H^{2,0} \oplus H^{1,1} \oplus H^{0,2} \arrow[ur, "\varphi"'] & \end{tikzcd}$$

where $\varphi : H^2(X, \mathbb{C}) \longrightarrow H^2(X, \mathcal{O}_X)$ is the projection to the (0, 2) - component. Hence we have $\alpha + \beta\tau \in \mathrm{Im}(c) = \ker(P)$ if and only if $p(\alpha + \beta\tau) = \alpha_{(0,2)} + \beta_{(0,2)}\tau = 0$. By taking complex conjugates, it is also equivalent to $\alpha_{(2,o)} + \beta_{(2,0)}\bar{\tau} = 0$. q.e.d.

The following example is due to A.Moriwaki.

Example(3.2.) (Moriwaki) Let X be a projective manifold which enjoys the following conditions.

(1) $h^{2,0}(X) = 1$.

(2) Let ω be a holomorphic 2 - form on X. Then there exists an automorphism g of X with $g^*\omega = t\omega$, $t \in \mathbb{C}$, $t \notin \mathbb{R}$.

(For example, put $X = E_\rho \times E_\rho$, $E_\rho \simeq \mathbb{C}/(1, \rho)$, $\rho = \exp(2\pi i/3)$) and $g : (x, y) \longrightarrow (\rho x, \rho y)$.)

Next, choose a transcendental cycle $\beta \in H^2(X, \mathbb{Z})$ arbitrarily and put $\alpha := g^*\beta \in H^2(X, \mathbb{Z})$. Since the action of $\langle g\rangle$ on $H^2(X, \mathbb{Z})$ is compatible with the Hodge decomposition, we have

$\alpha_{(2,0)} = t\beta_{(2,0)}$. Put $\tau := -\bar{t} \notin \mathbb{R}$ and $E \simeq \mathbb{C}/(1,\tau)$. Then by lemma (3.1.), there exists $\eta \in H^1(X, \mathcal{O}(E))$ with $c(\eta) = \alpha + \beta\tau \in H^2(X, G)$.

Put $Y := (X \times E)^{\eta}$. From our construction, the first Chern class $c(\eta) \in H^2(X, G)$ does not satisfy condition(A) in proposition(1.9.). Hence Y cannot be obtained from the trivial bundle over X by generalized logarithmic transformations.

Remark(3.3.) By theorem(A), example (2.9.) and example (3.2.), there exists a *non - Kähler* elliptic bundle $f : Y \longrightarrow X$ over a projective manifold X such that $Y|_C \longrightarrow C$ is Kähler for all non - singular curve $C \subset X$.

References.

[1] W.Barth, C.Peters and A.Van de Ven, Compact complex surfaces, Berlin, Heidelberg - New York - Tokyo, Springer, (1984).

[2] E.Calabi and B.Eckmann, A class of compact complex manifolds which are not algebraic, Ann. Math. 58 (1953), 494 - 500.

[3] A.Fujiki, On the structure of compact complex manifolds in $\mathscr{C}$, Advanced studies in Pure Math., 1, ed. S.Iitaka and H.morikawa, (1982), 229 - 300.

[4] ———, On automorphism groups of compact Kähler manifolds, Invent. math., 44 (1978), 225 - 258.

[5] Y.Fujimoto, Logarithmic transformations on elliptic fiber spaces, J. Math. Kyoto Univ., 28 - 1 (1988), 91 - 110.

[6] ———, Pluricanonical mappings of elliptic fiber spaces, Proc. Japan Acad, 62 (1986), 308 - 310.

[7] R.Harvey and H.Lawson, An intrinsic characterization of Kähler manifolds, Invent. math. 74 (1983),169 - 198.

[8] M.Kato, On compact complex threefolds with lines, Japan. J. Math. vol. 11, No.1, (1985), 1 - 58.

[9] T.Katsura and K.Ueno, On elliptic surfaces in characteristic p, Math. Ann. 272 (1985), 291 - 330.

[10] K.Kodaira, On the structure of compact complex analytic surfaces, II — III, Amer J. Math., 86 (1964), 751 - 798, 88 (1966), 682 - 721.

[11] ——— , On compact analytic surfaces, II — III, Annals of Math., 77 (1963), 563 - 626, 78 (1963), 1 — 40.

[12] Y.Miyaoka, Kähler metrics on elliptic surfaces,

Proc. Japan Acad., 50 (1974), 533 - 536.

[13] K.Ueno, Degeneration of elliptic surfaces and certain non - Kähler manifolds, Progress in Mathematics., 39, Birkhäuser - Boston (1983), 545 - 556.

[14] ———, On non - kähler degenerations of complex tori. Recent Trends in Mathematics, Reinhardsbrunn 1982, Teubner - Texte zur Math. 50 (1983).

[15] ———, On compact analytic threefolds with non - trivial Albanese tori, Math. Ann., 278 (1987), 41 - 70.

[16] J.Varouchas, Stabilité de la classe des variétés Kählериennes pour les certain morphisms propres, Invent. Math., 77 (1984), 117 - 128.

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A note on an example of a compactification of $\mathbb{C}^3$

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§0. Introduction. Let (X,Y) be a (projective) compactification of $\mathbb{C}^3$ with the second Betti number $b_2(X) = 1$. Then Y is an irreducible ample divisor on X and the canonical divisor is written as $K_X = -rY$ $(r > 0,\ r \in \mathbb{Z})$. In the case of r 2, the structure of (X,Y) is completely determined (cf. [1], [2], [3], [9]). In the case of $r = 1$, Y must be non-normal (cf. [3]). Moreover, in this case, by Peternell-Schneider [9], X is a Fano 3-fold of first kind of index one, genus 12 and Y is a non-normal hyperplane section of X. In the paper [3], the author proved that such a compactification (X,Y) really exists. In fact, $X = V'_{22}$ is a Fano 3-fold constructed by Mukai-Umemura [7], and $Y = H'_{22}$ is a non-normal hyperplane section of V'_{22} whose singular locus is a line in V'_{22}.

Now, in this paper, we will study the double projection of V'_{22} from the singular locus of H'_{22}, which is a line, and give a geometric structure of the compactification (V'_{22}, H'_{22}) of $\mathbb{C}^3$ (Theorem in §2).

§1. Mukai - Umemura's construction.

Let $\mathbb{C}[x,y]$ be the polynomial ring of two complex variables x and y. $SL(2,\mathbb{C})$ operates on $C[x,y]$ as follows:

$$\begin{cases} x^{\sigma} = ax + by \\ y^{\sigma} = cx + dy \end{cases}$$

for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$.

Let us denote by R_n the vector space of the homogeneous polynomials of degree n. Let $f(x,y) = \sum_{i=0}^{n} a_i \binom{n}{i} x^{n-i} y^i \in R_n$ be a non-zero homogeneous polynomial of degree n. We take $(a_0 : \dots : a_n)$ as homogeneous coordinates on the projective space $\mathbb{P}(R_n) \cong \mathbb{P}^n$ on which $SL(2,\mathbb{C})$ operates. Let us denote by $X(f)$ the closure $\overline{SL(2,\mathbb{C}).f}$ of the $SL(2,\mathbb{C})$-orbit $SL(2,\mathbb{C}).f$ of f in $\mathbb{P}(R_n)$. Then $SL(2,\mathbb{C})$ operates on $X(f)$. Now, let us consider the following two polynomials:

$$\begin{cases} f_6(x,y) = xy(x^4 - y^4) \\ h_{12}(x,y) = xy(x^{10} + 11x^5y^5 + y^{10}). \end{cases}$$

We put

$$\begin{cases} V_5 := X(f_6) \hookrightarrow \mathbb{P}(R_6) \cong \mathbb{P}^6 \\ V'_{22} := X(h_{12}) \hookrightarrow \mathbb{P}(R_{12}) \cong \mathbb{P}^{12} \end{cases}$$

Then,

Lemma 1 (Lemma 3.3 in [7]). (1) V_5 is a Fano threefold of index 2, genus 21 and the hyperplane section of $V_5 \hookrightarrow \mathbb{P}^6$ is the positive generator of $\operatorname{Pic} V_5 \cong \mathbb{Z}$.

(2) V_{22}' is a Fano threefold of index 1, genus 12 and the hyperplane section of $V_{22}' \hookrightarrow \mathbb{P}^{12}$ is the positive generator of Pic $V_{22}' \cong \mathbb{Z}$.

The defining equations for V_5, V_{22}' are the following (p.505-p.506 in [7]) :

$$(V_5) \qquad \begin{cases} a_0a_4 - 4a_1a_3 + 3a_2^2 = 0 \\ a_0a_5 - 3a_1a_4 + 2a_2a_3 = 0 \\ a_0a_6 - 9a_2a_4 + 8a_3^2 = 0 \\ a_1a_6 - 3a_2a_5 + 2a_3a_4 = 0 \\ a_2a_6 - 4a_3a_5 + 3a_4^2 = 0 \end{cases}$$

$$(V_{22}') \qquad \sum_{\lambda=0}^{\rho} \binom{8}{\lambda}\cdot\binom{8}{\rho-\lambda}(a_\lambda a_{\rho+4-\lambda} - 4a_{\lambda+1}a_{\rho+3-\lambda} + 3a_{\lambda+2}a_{\rho+2-\lambda})=0$$

$$(0 \leqq \rho \leqq 16)$$

Now, we put

$$H_5^\infty := V_5 \cap \{a_6=0\} \hookrightarrow P^5$$

$$H_{22}' := V_{22}' \cap \{a_0=0\} \hookrightarrow P^{11}$$

Let us denote by Sing H_5^∞ (resp. Sing H_{22}') the singular locus of H_5^∞ (resp. H_{22}'). Then we have

Lemma 2. (1)([2]). $V_5 - H_5^\infty \cong \mathbb{C}^3$ and $\mathrm{Sing}\, H_5^\infty =: s$ is a line in V_5 with the normal bundle $N_{s|V_5} \cong O(-1) \oplus O(1)$.

(2)([3]). $V'_{22} - H'_{22} \cong \mathbb{C}^3$ and $\mathrm{Sing}\, H'_{22} =: \ell$ is a line in V'_{22} with the normal bundle $N_{\ell|V'_{22}} \cong O(-2) \oplus O(1)$. In particular, there is no other line in V'_{22} which intersects the line ℓ .

By a direct computation, we have

Lemma 3. (1) $\mathrm{Sing}\, H'_{22} =: \ell = \{a_0 = a_1 = \ldots\ldots = a_{10} = 0\} \hookrightarrow \mathbb{P}^{12}$.

(2) $H'_{22} \cap \{a_1 \neq 0\} \cong \mathbb{C}^2$.

(3) $H'_{22} \cap \{a_{12} \neq 0\} \cong V(f) \hookrightarrow \mathbb{C}^3(x,y,z)$, where

$$(*) \quad f = b_0x^4 + (b_1yz + b_2z^3)x^3 + (b_3y^3 + b_4y^2z^2 + b_5yz^4)x^2$$
$$(b_6y^4z + b_7y^3z^3)x + b_8y^6 + b_9y^5z^2 \ ,$$

$$\left\{\begin{array}{l} b_0 = -2^8 5^2 \\ b_1 = 2^9 3^3 5 \\ b_2 = -2^6 3^4 5 \\ b_3 = -2^8 3^3 7 \\ b_4 = -2^4 3^4 127 \end{array}\right. \qquad \left\{\begin{array}{l} b_5 = 2^9 3^5 \\ b_6 = 2^2 3^6 89 \\ b_7 = -2^8 3^6 \\ b_8 = -3^6 5^3 \\ b_9 = 2^5 3^7 \end{array}\right.$$

(4) $\mathrm{Sing}\, V(f) = \{x = y = 0\} \cong \ell \cap \{a_{12} \neq 0\}$. Furthermore, the multiplicity $\mathrm{mult}_\ell Y = 3$, namely, Y is a unique element of the linear system $|O_{V'_{22}}(1) \otimes I_\ell^3|$, where I_ℓ is the ideal sheaf of ℓ in $O_{V'_{22}}$

§2. Double projection.

We will study the double projection of V'_{22} from the singular locus ℓ := Sing H'_{22} which is a line in V'_{22}. For simplicity, we put $X := V'_{22}$ and $Y := H'_{22}$.

(I) Let $\sigma_1 : X_1 \to X$ be the blowing up of X along the line ℓ and put $L_1 := \sigma_1^{-1}(\ell)$. By Lemma 2-(2), we have $L_1 \cong \mathbb{F}_3$ (Hirzebruch surface). Let Y_1 be the proper tranceform of Y in X_1. By Lemma 3-(4), we have a linear equivalence $Y_1 \sim \sigma_1^* H - 3L_1$.

Let us consider the linear systems $|H| := |\mathcal{O}_X(1) \otimes I_\ell^2|$ and $|H_1| := |\sigma_1^* H - 2L_1|$. Then we have

Lemma 4 (Lemma 5.4 in [4]). (1) $\dim_{\mathbb{C}}|H| = \dim_{\mathbb{C}}|H_1| = 6$ and $\dim_{\mathbb{C}}|\sigma_1^* H - 3L_1| = \dim_{\mathbb{C}}|Y_1| = 0$ (namely, Y_1 is a unique member of the linear system $|\sigma_1^* H - 3L_1|$.

(2) $(H_1^3) = 2$

(3) $Y_1 \cap L_1 \sim 3\ell_1 + 7f_1$ in L_1, where ℓ_1, f_1 is the negative section and a fiber of L_1.

Since $K_{X_1} \sim -\sigma_1^* H + L_1$ and $(L_1 . \ell_1) = 1$, we have $(K_{X_1} . \ell_1) = 0$. Thus, by the following exact sequence:

$$
\begin{array}{ccccccccc}
0 \to & N_{\ell_1|L_1} & \to & N_{\ell_1|X_1} & \to & N_{L_1|X_1}|\ell_1 & \to 0 \\
& \wr\| & & & & \wr\| & \\
& \mathcal{O}(-3) & & \mathcal{O}(a)\oplus\mathcal{O}(b) & & \mathcal{O}(1) &
\end{array}
,
$$

where $a + b = c_1(N_{\ell_1|X_1}) = -2$, we have

Lemma 5.

$$N_{\ell_1}|X_1 \cong \begin{cases} \text{(a)} & \mathcal{O}(-1) \oplus \mathcal{O}(-1) \\ \text{(b)} & \mathcal{O}(-2) \oplus \mathcal{O} \\ \text{(c)} & \mathcal{O}(-3) \oplus \mathcal{O}(1) \end{cases}$$

Lemma 6. $\mathrm{Bs}\ H_1 = \ell_1$.

Proof. Since $(H_1 . \ell_1) = -1 < 0$, $\ell_1 \subsetneqq \mathrm{Bs}\ H_1$. By Lemma 2-(2), there is no other line in X which intersects ℓ . Thus, by the same argument as in the proof of Lemma 5.4-(ii) in [4], we have the claim.

Let us denote by $\pi_{2\ell}$ a rational map defined by the linear system H , which is called the double projection from ℓ . By Lemma 4-(1), we have a diagram :

$$\begin{tikzcd} X_1 \arrow[d, "\sigma_1"'] \arrow[dr, dashed, "\Phi_1"] & \\ X \arrow[r, dashed, "\pi_{2\ell}"'] & \mathbb{P}^6 \end{tikzcd}$$

where $\Phi_1 := \Phi_{H_1}$ is a rational map defined by the linear system H_1 .

We will resolve the indeterminancy of the rational map $\Phi_1 : X_1 \dashrightarrow \mathbb{P}^6$. For this, we need the following

Lemma 7. (1) $\mathrm{Sing}\ Y_1 = 2\ell_1$, namely, $\mathrm{Sing}\ Y_1 = \ell_1$ as a set and $\mathrm{mult}_{\ell_1} Y_1 = 2$.

(2) $Y_1 \cap L_1 = A_1 + A_2 + A_3$, where $A_1 \sim 2\ell_1, A_2 \sim \ell_1 + 4f_1$, $A_3 \sim 3f_1$ in L_1 .

Proof. One can obtain easily the (local) defining equation of Y_1 in X_1 from (*) in Lemma 3-(3). Looking at this equation, we have the assertion.

(II) Let $\sigma_2 : X_2 \to X_1$ be the blowing up of X_1 along the negative section ℓ_1 of L_1 and put $L_2 := \sigma_2^{-1}(\ell_1)$. By Lemma 5, we have the following three cases:

$$L_2 \cong \begin{cases} (a) & \mathbb{P}^1 \times \mathbb{P}^1 \\ (b) & \mathbb{F}_2 \\ (c) & \mathbb{F}_4 \end{cases}$$

Let Y_2 be the proper transform of Y_1 in X_2. By Lemma 7-(1) we have $Y_2 \sim \sigma_2^* Y_1 - 2L_2$. We put $H_2 := \sigma_2^* H_1 - L_2$. Let ℓ_2, f_2 be the non-positive section and a fiber of L_2. Since

$$(L_2^2) \sim \begin{cases} -\ell_2 - f_2 & \text{if } L_2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \\ -\ell_2 - 2f_2 & \text{if } L_2 \cong \mathbb{F}_2 \\ -\ell_2 - 3f_2 & \text{if } L_2 \cong \mathbb{F}_4 \end{cases},$$

we have easily

Lemma 8.

$$Y_2 \cap L_2 \sim \begin{cases} (a) & 2\ell_2 & \text{if } L_2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \\ (b) & 2\ell_2 + 2f_2 & \text{if } L_2 \cong \mathbb{F}_2 \\ (c) & 2\ell_2 + 4f_2 & \text{if } L_2 \cong \mathbb{F}_4 \end{cases}$$

On the other hand, by a direct computation, we have

Lemma 9. (1) $Y_2 \cap L_2 = B_1 + B_2$, where $B_1 \sim 2\ell_2$, $B_2 \sim 2f_2$ in L_2.

(2) $\operatorname{Sing} Y_2 = 2\ell_2$.

Corollary 10. $L_2 \cong F_2$, namely, $N_{\ell_1|X_1} \cong O(-2) \oplus O$.

Lemma 11. $\mathrm{Bs}|H_2| = \ell_2$

Proof. Since $(H_2\ \ell_2) = -1$, $\ell_2 \subsetneqq \mathrm{Bs}\ H_2$. On the other hand, since $|H_2| \cap L_2 \subsetneqq |\ell_2 + f_2|$, we have the claim.

Since $(K_{X_2} \cdot \ell_2) = 0 = (L_2 \cdot \ell_2)$, by an exact sequence

$$
\begin{array}{ccccccc}
0 \to N_{\ell_2|L_2} & \to & N_{\ell_2|X_2} & \to & N_{L_2|X_2|\ell_2} & \to 0 \\
\wr\| & & \wr\| & & \wr\| & \\
O(-2) & \to & O(a)\oplus O(b) & \to & O & ,
\end{array}
$$

where $a + b = -2$, we have

Lemma 12.

$$
N_{\ell_2|X_2} \cong \begin{cases} \text{(a)} & O(-1) \oplus O(-1) \\ \text{(b)} & O(-2) \oplus O \end{cases}
$$

(III) Let $\sigma_3 : X_3 \to X_2$ be the blowing up of X_2 along the section ℓ_2 of L_2 and put $L_3 = \sigma_3^{-1}(\ell_2)$. By Lemma 11, we have the two cases:

$$
L_3 \cong \begin{cases} \text{(a)} & \mathbb{P}^1 \times \mathbb{P}^1 , \\ \text{(b)} & \mathbb{F}_2 \end{cases}
$$

Let Y_3 be the proper transform of Y_2 in X_3. By Lemma 9-(2), we have $Y_3 \sim \sigma_3^*(Y_2) - 2L_3$. We put $H_3 := \sigma_3^* H_2 - L_3$. Let ℓ_3, f_3 be the non-positive section and a fiber of L_3 . Since

$$
(L_3^2) \sim \begin{cases} \text{(a)} & -\ell_3 - f_3 \ \text{ if } \ L_3 \cong P^1 \times P^1 \\ \text{(b)} & -\ell_3 - 2f_3 \ \text{ if } \ L_3 \cong F_2 , \end{cases}
$$

Lemma 12.

$$Y_3 \cap L_3 \cong \begin{cases} \text{(a)}\ 2\ell_3 & \text{if } L_3 \cong P^1 \times P^1 \\ \text{(b)}\ 2\ell_3 + 2f_3 & \text{if } L_3 \cong F_2 . \end{cases}$$

By a direct computation, one can easily have the following

Lemma 13.(1) $Y_3 \cap L_3 = C_1 + C_2$, where $C_1 \sim 2\ell_3$, $C_2 \sim 2f_3$.

(2) $\operatorname{Sing} Y_3 = 2\ell_3 + 2f_3$.

Corollary 14. $L_3 \cong F_2$,namely, $N_{\ell_3|X_3} \cong O(-2) \oplus O$.

Moreover, we have

Lemma 15. $\operatorname{Bs}|H_3| = \ell_3$.

Lemma 16.

$$N_{\ell_3|X_3} \cong \begin{cases} \text{(a)}\ O(-1) \oplus O(-1) \\ \text{(b)}\ O(-2) \oplus O \end{cases},$$

The proof is similar to that of Lemma 11, Lemma 12.

(IV) Let $\sigma_4 : X_4 \to X_3$ be the blowing up of X_3 along the section ℓ_3 of L_3 and put $L_4 := \sigma_4^{-1}(\ell_3)$. Let Y_4 be the proper transform of Y_3 in X_4 and put $H_4 := \sigma_4^* H_3 - L_4$ and let ℓ_4 , f_4 be the zero section , a fiber of L_4. By Lemma 16, wehave the two cases:

$$L_4 \cong \begin{cases} \text{(a)}\ \mathbb{P}^1 \times \mathbb{P}^1 \\ \text{(b)}\ \mathbb{F}_2 \end{cases}.$$

By Lemma 13-(2), we have $Y_4 \sim \sigma_4^* Y_3 - 2L_4$. Since

$$(L_4^2) \sim \begin{cases} \text{(a)}\ -\ell_4 - f_4 & \text{if } L_4 \cong \mathbb{P}^1 \times \mathbb{P}^1 \\ \text{(b)}\ -\ell_4 - 2f_4 & \text{if } L_4 \cong \mathbb{F}_2, \end{cases}$$

we have

Lemma 17.

$$Y_4 \cap L_4 \sim \begin{cases} \text{(a)} \quad 2\ell_4 & \text{if} \quad L_4 \cong \mathbb{P}^1 \times \mathbb{P}^1 \\ \text{(b)} \quad 2\ell_4 + 2f_4 & \text{if} \quad L_4 \cong \mathbb{F}_2 . \end{cases}$$

On the other hand, by a direct computation, we have

Lemma 18.

$$Y_4 \cap L_4 = D \sim 2\ell_4 \quad \text{in} \quad L_4.$$

Corollary 19.

$$L_4 \cong \mathbb{P}^1 \times \mathbb{P}^1, \text{ namely, } \quad N_{\ell_3 | X_3} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

Moreover, we have

Lemma 20. (1) $\mathrm{Bs}|H_4| = \phi$.

(2) $(H_4^3) = 5$.

(3) $(H_4^2 \cdot Y_4) = (H_4^2 \cdot L_4) = (H_4^2 \cdot L_j^{(4)}) = 0$ ($j = 2,3$) , and $(H_4^2 \cdot L_1^{(4)}) = 5$, where $L_j^{(4)}$ is the proper transform of L_j in X_4 .

(4) $(H_4 \cdot A_2^{(4)}) = 5$, $(H_4 \cdot A_3^{(4)}) = 0$, $(H_4 \cdot f_1^{(4)}) = 1$, where $A_j^{(4)}$, $f_1^{(4)}$ are the proper transforms of A_j and a general fiber f_1 in X_4 .

By Lemma 20-(1), we have the morphism $\Phi : X_4 \to \mathbb{P}^6$, defined by the linear system $|H_4|$. We put $W := \Phi(X_4)$. By Lemma 20-(2), we have $\deg W = 5$. By construction, W is a compactification of $\mathbb{C}^3$ with $b_2(W) = 1$.

Since $N_{\ell_3|X_3} \cong O(-1) + O(-1)$, by Reid [8] , L_4 can be blown down along ℓ_4, and then blowing downs can be done step-by-step. Finally, we have a smooth projective threefold W_1 with $b_2(W_1) = 2$, morphism $\Phi_2 \colon X_4 \to W$, $\Phi_1 : W_1 \to W$, and a birational map $\rho \colon X_1 \to W_1$ which is called a flop such that

(i) $\Phi = \Phi_1 \Phi_2$

(ii) $X_1 - \ell_1 \overset{\rho}{\cong} W_1 - g_1$, where $g_1 := \Phi_2(L_4 \cap L_1^{(4)})$.

(D)
$$\begin{tikzcd}
& X_4 \arrow[dl] \arrow[dr, "\Phi_2"] & & \\
X_1 \arrow[rr, dashed, "\rho"] \arrow[d, "\sigma_1"'] & & W_1 \arrow[d, "\Phi_1"] & \\
X \arrow[rr, dashed] & & W \arrow[r, hook] & \mathbb{P}^6
\end{tikzcd}$$

(iii) $W_1 - E_1 \cong W - \Gamma$, where $E_1 = \rho(Y_1), \Gamma = \Phi(Y_4)$ $= \Phi_1(Y_1)$.

Since $-K_{X_1} = Y_1 + 2L_1$ and $\rho \colon X_1 - \ell_1 \cong W_1 - g_1$, we have $-K_{W_1} = E_1 + 2F_1$, where $F_1 = \rho(L_1)$. We remark that $\Gamma = \Phi_1(E_1)$.

We put $Z = \Phi_1(F_1)$. By Lemma 20-(4), Z is swept out by lines in $W \hookrightarrow \mathbb{P}^6$ and $\Gamma \hookrightarrow Z$, $\Gamma \cap g = \{\text{one point}\}$,where $\deg \Gamma = 5$ and Γ is a smooth rational curve.

Let γ be a conic in X which intersects the line ℓ . Then, we have $\gamma \hookrightarrow Y$. Let γ_1 be the proper transform of γ in X_1. We put $\widehat{\gamma_1} := \rho(\gamma_1)$. Since $(K_{W_1} \cdot \widehat{\gamma_1}) = -(E_1 \cdot \widehat{\gamma_1}) - 2(F_1 \cdot \widehat{\gamma_1}) = 1 - 2$ $= -1$, by K.M.M.[5], $\Phi_1 \colon W_1 \to W$ is the contraction of an exremal ray. Since the exceptional divisor E_1 is contracted to a smooth curve, by Mori [6], W is smooth. Thus $-K_W = 2Z$. Since $W - Z \cong \mathbb{C}^3$ by construction, Z is ample, namely, W is a Fano threefold of first kind of index 2 , genus 21. Since Z is swept out by lines on W, Z is non-normal. Therefore, we have $(W,Z) \cong (V_5, H_5)$ (see §1) (cf.[2]). Thus we have

Theorem. There is a birational map $\pi_{2\ell}: V'_{22} \dashrightarrow V_5$, called the double projection from the line ℓ , such that the restriction $\pi_{2\ell} : V'_{22} - H'_{22} \xrightarrow{\sim} V_5 - H_5^{\infty}$ ($\cong \mathbb{C}^3$) is isomorphic. The inverse map $\rho_2^{-1} : V_5 \dashrightarrow V'_{22}$ is given by the linear system $|\mathcal{O}_{V_5}(3) \otimes I_{\Gamma}^2|$.

References

[1] M. Furushima, Singular del Pezzo surfaces and analytic compactifications of 3-dimensional complex affine space $\mathbb{C}^3$, Nagoya Math. J. 104 (1986), 1-28.

[2] M. Furushima-N. Nakayama, The family of lines on the Fano threefold V_5 , Nagoya Math. J. 116 (1989), 111-122.

[3] M. Furushima, Analytic compactifications of $\mathbb{C}^3$, to appear in Compositio Math.

[4] V.A. Iskovskih, Fano 3-folds II, Math. U.S.S.R. Izvestija 12 (1978), 469-506.

[5] Y.Kawamata-K.Matsuda-M.Matsuki, Introduction to the minimal model problem, in Algebraic Geometry, Sendai, (T.Oda,ed.) Advance Studies in Pure Math. 10 (1987), Kinokuniya, Tokyo and North-Holland, Amsterdam, 551-590.

[6] S. Mori, Threefolds whose canonical bundle are not numerical

effective, Ann. Math. 116 (1982), 133-176.

[7] S.Mukai-H.Umemura, Minimal rational threefolds, Lecture note in Math. 1016, Springer-Verlag (1983), 490-518.

[8] M.Reid, Minimal models of canonical 3-folds, in Algebraic Varieties and Analytic Varieties (S.Iitaka, ed.), Advance Studies in Pure Math. 1 (1983), Kinokuniya, Tokyo and North-Holland, Amsterdam, 131-180.

[9] Th.Peternell-M.Schneider, Compactifications of $\mathbb{C}^3$.I, Math. Ann. 280 (1988), 129-146.

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Algebraic Surfaces of General Type with $c_1^2 = 3p_g - 6$

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Introduction.

Let S be a minimal algebraic surface of general type defined over the complex number field $\mathbf{C}$. Castelnuovo's second inequality says that the canonical map Φ_K of S can be birational onto its image only when $c_1(S)^2 \geq 3p_g(S) - 7$ (see, e.g. [11, II, Lemma(1.1)]). Surfaces with $c_1^2 = 3p_g - 7$ have been studied by several authors (e.g. [8], [2]). The purpose of the present article is to determine, to some extent, the structure of minimal surfaces with $c_1^2 = 3p_g - 6$. We remark that Horikawa ([11, III and IV]) studied, among others, those with $(p_g, c_1^2) = (3, 3)$, $(4, 6)$ in detail. So we restrict ourselves to the case $p_g \geq 5$.

In §1, we classify them into three types according to the nature of Φ_K which is generically finite onto its image. Namely, a minimal surface of general type with $c_1^2 = 3p_g - 6$ is said to be of type I, type II or of type III, if $\deg \Phi_K = 1, 2$ or 3, respectively. We also state a result on regular surfaces of type II similar to one in [2]. In §2, we study type III surfaces S and show $p_g(S) \leq 5$, $q(S) = 0$. Those with $p_g = 3$, 4 are known to exist (see, [11, III and IV]). We show that S with $p_g = 5$ also exists. It has a pencil $|D|$ of nonhyperelliptic curves of genus 3 with one base point, and the canonical map induces on D the projection from it to a line, if we identify D with a plane quartic.

The rest of the article, §§3–9, is devoted to surfaces of type I. Please recall that the canonical image of a type I surface with $c_1^2 = 3p_g - 7$ is contained in a threefold of Δ-genus 0 [2]. An analoguos phenomena can be observed also in the present case. We show that the canonical image of a type I surface is contained in a threefold W of Δ-genus ≤ 1, which is, if $p_g \geq 6$, cut out by all quadrics through the canonical image (Theorem 3.1). As in the case of $c_1^2 = 3p_g - 7$, the proof is based on Castelnuovo's idea: We cut the canonical image twice by hyperplanes and count the number of quadrics through the resulting set of points. The key is a more recent result of Harris-Eisenbud [9] on the Hilbert functions of a special set of points in a projective space, which itself is a generalization of classical Castelnuovo's Lemma. We remark that varieties of Δ-genus ≤ 1 are successfully classified by Fujita ([4], [5], [7]). This enables us to study type I surfaces as divisors on "known" threefolds, and to clarify their structure. We show, among others, every surface of type I has a pencil of nonhyperelliptic curves of genus 3 if $p_g \geq 12$ (Theorem 6.2).

As mentioned above, surfaces of type I are further classified into two types by the Δ-genus of the threefold W. We say that S is of type I-0 or type I-1, according to whether W is of Δ-genus 0 or 1. In §4, we study surfaces of type I-1. We show Theorem 4.2, which says in particular that they satisfy $p_g \leq 11$. In §§5–9, we study surfaces of type I-0. In §5, we discuss whether the canonical map can be lifted to a holomorphic map into a nonsingular model of W, and show Proposition 5.6. In §6, we determine the linear equivalence class on W of the canonical image, and prepare some Lemmas for the later use. It will be clarified that every surface of type I-0 has a pencil of nonhyperelliptic curves of genus 3 or 4. In the last case, we have $p_g \leq 7$. The existence of those with $p_g = 7$ and $p_g = 6$ is shown in §7 and §8, respectively. In §9, we study the case of genus 3. We sketch a member of the pencil arising from the double curve of the canonical image. In general, it is a hyperelliptic curve of genus 3.

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Contents

1 Canonical maps.

1.1 Let S be a minimal algebraic surface of general type defined over the complex number field $\mathbf{C}$. We denote by $p_g(S)$, $q(S)$ and $c_1(S)^2$ the geometric genus, the irregularity and the Chern number of S, respectively. The canonical divisor of S will be denoted by K. We let $\sigma : \widetilde{S} \to S$ be the composition of quadric transformations which is the shortest among those with the property that the variable part $|L|$ of $|\sigma^* K|$ is free from base points. We denote by $\widetilde{K}$ and E the canonical divisor of $\widetilde{S}$ and the exceptional divisor of σ, respectively. In particular, we have $\widetilde{K} \sim \sigma^* K + E$, where the symbol $\sim$ means the linear equivalence of divisors.

In what follows, we often use the standard fact that, if a surface admits a rational map of degree less than three onto a ruled surface, then its canonical map cannot be birational onto its image.

We recall a well-known result due to Castelnuovo (see, [9]).

Lemma 1.2 (Castelnuovo's bound)

Let $C \subset \mathbf{P}^r$ be an irreducible nondegenerate curve of degree d and geometric genus $g(C)$. Then

$$g(C) \leq \pi_0(d,r) = \binom{m_0}{2}(r-1) + m_0\epsilon_0$$

where m_0 and ϵ_0 are nonnegative integers satisfying

$$d = m_0(r-1) + \epsilon_0 + 1,\ 0 \leq \epsilon_0 \leq r-2.$$

Lemma 1.3 *Let S be a minimal algebraic surface of general type with $c_1(S)^2 = 3p_g(S) - 6$. Let $\Phi_K : S \to S_0 \subset \mathbf{P}^{p_g-1}$ be the canonical map of S. Then one of the following occurs:*

1) *Φ_K induces a birational holomorphic map onto S_0.*

2) *Φ_K induces a generically finite rational map of degree 2 onto S_0 which is birationally equivalent to a ruled surface.*

3) *Φ_K induces a holomorphic map of degree 3 onto S_0 which is a surface of minimal degree $p_g - 2$ in $\mathbf{P}^{p_g-1}$.*

Proof. We remark that $|K|$ is not composite with a pencil by [11, III, Theorem1.1] and [3, p. 136]. Thus Φ_K induces a generically finite rational map onto its image. Let $\sigma : \widetilde{S} \to S$ be as in 1.1. Since S_0 is an irreducible nondegenerate surface in $\mathbf{P}^{p_g-1}$, we see from [10, Lemma1] that

$$3p_g - 6 = c_1^2 \geq L^2 = (\deg \Phi_K)(\deg S_0) \geq (\deg \Phi_K)(p_g - 2).$$

So we have $\deg \Phi_K \leq 3$. Furthermore, if $\deg \Phi_K = 3$, the equality holds everywhere. This implies that S_0 is a surface of minimal degree and that $|K|$ is free from base points [10, Lemma 1]. Thus we have 3). If $\deg \Phi_K = 2$, then we are in the case 2) and the statement for S_0 can be found in [3]. We show that $|K|$ is free from base points if Φ_K is birational. For this purpose, we recall the inequality $L^2 \geq 3p_g - 7$ ([11, II, Lemma(1.1)]). Thus we have either

i) $c_1^2 = L^2$, or

ii) $c_1^2 = L^2 + 1$.

We show that ii) leads us to a contradiction. Assume that ii) is the case. Then, by the argument in [10, §1], we see that $|K|$ has the unique base point P and σ is the blowing-up of S at P. Thus we have $|\sigma^* K| = |L| + E$, where E is the exceptional (-1)-curve and $LE = 1$. Let C be a general member of $|L|$, which we can assume irreducible and nonsingular. Since $C_0 := \Phi_K(C)$ is an irreducible nondegenerate curve of degree $3p_g - 7$ in $\mathbf{P}^{p_g-2}$, Castelnuovo's bound (Lemma 1.2) shows that its geometric genus $g(C)$ is bounded from above by $\pi_0(3p_g - 7, p_g - 2) = 3p_g - 6$. On the other hand, by the virtual genus formula, we have

$$g(C) = \frac{1}{2}C(\widetilde{K} + C) + 1 = L^2 + LE + 1 = 3p_g - 5.$$

Thus $\pi_0(3p_g - 7, p_g - 2) < g(C)$, a contradiction. If i) is the case, the absence of base points of $|K|$ follows from [10, Lemma 1]. *q.e.d.*

Definition 1.4 Let S be as in Lemma 1.3. We say that it is of type I, type II or type III if Φ_K satisfies 1), 2) or 3) in Lemma 1.3, respectively.

Throughout the paper, we assume that $p_g \geq 5$, since the cases $p_g = 3$, 4 can be found in [11, III and IV].

We now consider type II surfaces. Since we shall show in Lemma 2.2 and Theorem 3.1 that surfaces of types I and III are regular, we restrict ourselves to regular surfaces of type II here. Then Φ_K and the ruling of S_0 induce on S a linear pencil of hyperelliptic curves. Though the structure is simple, it seems hard to classify them completely. As for regular surfaces of type II, we have the following:

Proposition 1.5 *Let S be a minimal algebraic surface of general type with $c_1^2 = 3p_g - 6$ and $q = 0$ which is of type* II *in the sense of* 1.4. *If $p_g > 50$, then it has a unique pencil of hyperelliptic curves of genus less than* 5. *Conversely, take $g \in \{2, 3, 4\}$ and fix it. Then, for any pair of integers (x, y) satisfying $y = 3x - 6$ and $x \geq 5$, there exists a minimal regular surface S of type* II *with a pencil of hyperelliptic curves of genus g such that $p_g(S) = x$, $c_1(S)^2 = y$.*

Proof. The first half follows from a result of [13]. The last half can be shown by the method given in [2]. *q.e.d.*

2 Surfaces of type III.

In this section, we study surfaces of type III. Those with $p_g = 3, 4$ exist (see [11, II and III]). Among others, we shall use the following notation. For any nonnegative integer d, we denote by Σ_d the Hirzebruch surface of degree d. We let Γ denote a fiber of Σ_d and Δ_0 a section with $\Delta_0^2 = -d$.

2.1 As we have seen in Lemma 1.3, the canonical image of a type III surface is a surface of minimal degree. Here we recall a result of [12] (see also [4]). Let V be an irreducible nondegenerate surface of minimal degree $n-1$ in $\mathbf{P}^n$. Then it is either

a) $\mathbf{P}^2$ $(n=2)$,

b) $\mathbf{P}^2$ embedded into $\mathbf{P}^5$ by the complete linear system of quadrics $(n=5)$,

c) Σ_d embedded into $\mathbf{P}^n$ by $|\Delta_0 + \frac{n-1+d}{2}\Gamma|$, where $n-d-3$ is a nonnegative even integer $(n \geq 3)$, or

d) a cone over a rational curve of degree $n-1$ in $\mathbf{P}^{n-1}$, that is, the image of Σ_{n-1} by $|\Delta_0 + (n-1)\Gamma|$ $(n \geq 3)$.

Lemma 2.2 *Every type* III *surface S satisfies $p_g(S) \leq 5$ and $q(S) = 0$. If $p_g(S) = 5$, then its canonical image is a cone over a rational curve of degree* 3.

Proof. We put $n = p_g - 1$. We assume $n \geq 5$ and show that this leads us to a contradiction. We denote by $f : S \to S_0$ the holomorphic map induced by Φ_K. Then f is of degree 3 and S_0 is one of the surfaces in 2.1. Since $n \geq 5$, we need not consider the case a). If b) is the case, then we have $K \sim 2f^*l$, where l is a line in $\mathbf{P}^2$. Thus $(f^*l)(K+f^*l) = 3(f^*l)^2 = 9$. This contradicts that $(f^*l)(K+f^*l)$ is even. Similarly, if c) is the case, we have $K \sim f^*(\Delta_0 + ((n-1+d)/2)\Gamma)$ and get a contradiction because $(f^*\Gamma)(K+f^*\Gamma) = 3$.

We consider the case d). Then, by the same argument as in the proof of [11, I, Lemma 1], we have $K \sim (n-1)D + G$, where $|D|$ is a pencil and G is an effective (possibly zero) divisor with $KG = 0$. In particular, we have $G^2 \leq 0$ by Hodge's index theorem. Since $3(n-1) = K^2 = K((n-1)D+G) = (n-1)KD$, we get $KD = 3$. Thus we have

i) $0 = KG = (n-1)GD + G^2$, and

ii) $3 = KD = (n-1)D^2 + GD$.

Further, D^2 is a positive odd integer, because $D(K+D) = 3 + D^2$ is even. Since $G^2 \leq 0$, we have $GD \geq 0$ by i). From this and ii), we get $3 \geq (n-1)D^2 \geq n-1$, which contradicts the assumption $n \geq 5$.

Thus we have $p_g(S) \leq 5$ for a type III surface S. Then the vanishing of $q(S)$ follows from [11, V, Theorem 4.1]. *q.e.d.*

Theorem 2.3 *Let S be a type* III *surface with $p_g(S)=5$. Then S is the minimal nonsingular model of a surface S' defined by*

$$w^3+\alpha_1\zeta w^2+\alpha_2\zeta^2 w+\alpha_3\zeta^2=0 \tag{1}$$

in the total space of the line bundle $[2\Delta_0+4\Gamma]$ on Σ_3, where w is the fiber coordinate and

$$\alpha_j\in H^0(\Sigma_3,\mathcal{O}((j+[j/3])\Delta_0+4j\Gamma)),\ 1\le j\le 3,$$
$$\zeta\in H^0(\Sigma_3,\mathcal{O}(\Delta_0)).$$

Proof. This is a verbatim translation of [11, III, §4]. By the proof of Lemma 2.2, we have a pencil $|D|$ with $K\sim 3D$, $KD=3$ and $D^2=1$. Thus it has the unique base point P. We let $\sigma:\hat{S}\to S$ be the blowing-up with center P and put $E=\sigma^{-1}(P)$. Then the variable part $|\hat{D}|$ of $|\sigma^* D|$ defines a holomorphic map $g:\hat{S}\to \mathbf{P}^1$. Since $|K|$ has no base point, there exists $\eta\in H^0(\hat{S},\mathcal{O}(\sigma^*K))$ which does not vanish on E. Further, we can take nonzero $\omega\in H^0(\hat{S},\mathcal{O}(3E))$. Then the pair $(\omega,\ \eta)$ defines a holomorphic map $h:\hat{S}\to\Sigma_3$ with $h^*\Delta_0=3E$ and $\hat{K}\sim h^*(\Delta_0+3\Gamma)+E$, where $\hat{K}$ is the canonical divisor of $\hat{S}$. By the Riemann-Roch theorem,

$$\chi(\hat{K}+\hat{D})=\chi(K+D)=D(K+D)/2+\chi(\mathcal{O}_S)=8.$$

Since $H^p(-D)=0$ for $p<2$ by Ramanujam's vanishing theorem, we get $h^0(\hat{K}+\hat{D})=h^0(K+D)=8$. A nonzero $\xi\in H^0(2E)$ defines an injection $H^0(\hat{K}+\hat{D})\to H^0(\hat{K}+\hat{D}+2E)$. We have $\hat{K}+\hat{D}+2E\sim h^*(2\Delta_0+4\Gamma)$ and $h^0(\Sigma_3,\mathcal{O}(2\Delta_0+4\Gamma))=7$. Thus there exists $\varphi\in H^0(\hat{K}+\hat{D})$ such that $\varphi\xi$ is not induced by $H^0(\Sigma_3,\mathcal{O}(2\Delta_0+4\Gamma))$. We let W denote the total space of the line bundle $[2\Delta_0+4\Gamma]$ on Σ_3 and w its fiber coordinate. Then putting $w=\varphi\xi$, we get a holomorphic map $\hat{h}:\hat{S}\to W$ over h.

Put $S'=\hat{h}(\hat{S})$. Since h is of degree 3, we see that S' is birational to $\hat{S}$ by the choice of φ. We consider the subspace of $H^0(\hat{S},\mathcal{O}(h^*(6\Delta_0+12\Gamma)))$ consisting of those sections vanishing on $6E$. It contains

$$\begin{array}{ll}\varphi^3\xi^3, & \\ \varphi^2\xi^2\alpha\zeta & \text{with } \alpha\in H^0(\Sigma_3,\mathcal{O}(\Delta_0+4\Gamma)),\\ \varphi\xi\beta\zeta^2 & \text{with } \beta\in H^0(\Sigma_3,\mathcal{O}(2\Delta_0+8\Gamma)),\\ \gamma\zeta^2 & \text{with } \gamma\in H^0(\Sigma_3,\mathcal{O}(4\Delta_0+12\Gamma)).\end{array}$$

These represent 61 sections. On the other hand, since $h^*(6\Delta_0+12\Gamma)-6E\sim 4\sigma^*K$, we have $H^0(\hat{S},\mathcal{O}(h^*(6\Delta_0+12\Gamma)-6E))=60$. Thus S' is defined by the equation of the form (1). Then we can show, as in [11, III, §4], that S' has a double curve along $w=\zeta=0$ but other singular points are at most rational double points, and that $\hat{S}$ is the minimal resolution of S'. *q.e.d.*

3 Quadrics through canonical surfaces.

Recall that, for a projective variety $X \subset \mathbf{P}^n$ with $h^0(\mathcal{O}_X(1)) = n+1$, the Δ-genus is defined by $\Delta = \dim X + \deg X - n - 1$. We refer the reader to Fujita [4], [5], [7] for the theory of Δ-genus.

The purpose of this section is to prove the following:

Theorem 3.1 *Let S be a surface of type* I *with $p_g(S) \geq 5$. Then the irregularity $q(S)$ vanishes. The canonical image S_0 of S is contained in an irreducible nondegenerate variety W_0 of dimension* 3 *and Δ-genus ≤ 1. W_0 is cut out by all quadrics through S_0 unless it is a hypercubic in $\mathbf{P}^4$. Further, the following hold.*

1) *If W_0 is of Δ-genus* 0, *then it is a rational normal scroll.*

2) *If W_0 is of Δ-genus* 1, *then S_0 is projectively mormal and has only rational double points (RDP for short) as its singularity.*

Remark 3.2 Prof. E. Horikawa has a result similar to Theorem 3.1 (unpublished). His proof is based on a detailed study of quadrics through S_0 which is quite similar to "Petri's analysis".

3.3 For any projective variety $X \subset \mathbf{P}^r$, we denote by $\mathcal{I}_X$ the ideal sheaf of X in $\mathbf{P}^r$. We consider the cohomology exact sequence derived from

$$0 \to \mathcal{I}_X(n) \to \mathcal{O}_{\mathbf{P}^r}(n) \to \mathcal{O}_X(n) \to 0,$$

for any nonnegative integer n. The *Hilbert function* h_X of X is given by

$$h_X(n) = \dim_{\mathbf{C}} \operatorname{Im}\left\{H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(n)) \to H^0(X, \mathcal{O}_X(n))\right\}.$$

If Y is a general hyperplane section of X, then it follows from [9, Lemma(3.1)] that

$$\delta h_X(n) := h_X(n) - h_X(n-1) \geq h_Y(n),\ n \geq 1. \tag{2}$$

We remark that X is projectively normal if $\delta h_X(n) = h_Y(n)$ holds for any $n \in \mathbf{N}$.

We return to the situation we are interested in. Let S be a type I surface with $p_g \geq 5$ and put $r = p_g - 2$. We choose a general member $C \in |K|$. Since $|K|$ has no base points, we can assume that C is irreducible and nonsingular. Then the geometric genus $g(C)$ of C is given by $g(C) = \frac{1}{2}C(K+C) + 1 = K^2 + 1 = 3r + 1$. Since $2K|_C$ is the canonical divisor of C, we have

$$h^0(C, \mathcal{O}(nK|_C)) = \begin{cases} 3r+1 & \text{if } n = 2, \\ 3r(n-1) & \text{if } n \geq 3. \end{cases} \tag{3}$$

If we put $C_0 = \Phi_K(C)$, then C_0 is an irreducible nondegenerate curve of degree $K^2 = 3r$ in $\mathbf{P}^r$. We remark that it is linearly normal because $\pi_0(3r, r+1) < g(C)$ (see, Lemma 1.2). We take a general hyperplane section Z_0 of C_0. It is a set of $3r$ distinct points in $\mathbf{P}^{r-1}$ and satisfies $h_{Z_0}(1) = r$. Further, since it enjoys the *uniform position property*, it follows from [9, Corollary(3.5)] that

$$h_{Z_0}(n+1) \geq \min\{3r, h_{Z_0}(n) + r - 1\}, \quad n \in \mathbf{N}. \tag{4}$$

Thus we have (for $r \geq 4$)

$$\begin{array}{ll} h_{Z_0}(1) = r, & \\ h_{Z_0}(n) \geq n(r-1)+1, & \text{if } 2 \leq n \leq 3, \\ h_{Z_0}(n) = 3r, & \text{if } n \geq 4. \end{array}$$

Since C_0 *is nondegenerate*, we have $h_{C_0}(1) = r+1$. Thus (2) and (3) yield

$$3r + 1 = h^0(C, \mathcal{O}(2K|_C)) \geq h_{C_0}(2) \geq r + 1 + h_{Z_0}(2).$$

From this and (4), we get $h_{Z_0}(2) = 2r$ or $2r - 1$.

We now recall two surprizing results due to Castelnuovo and Harris-Eisenbud, respectively. For the proof, see [9, p. 106].

Lemma 3.4 (Castelnuovo)
If $Z \subset \mathbf{P}^{r-1}$ is a set of $d \geq 2r+1$ points in general position, then $h_Z(2) = 2r - 1$ holds if and only if Z lies on a rational normal curve R of degree $r-1$, cut out by all quadrics containing Z.

Lemma 3.5 (Harris-Eisenbud)
Let $Z \subset \mathbf{P}^{r-1}$, $r \geq 3$, be any finite set of $d \geq 2r+3$ points in uniform position with $h_Z(2) = 2r$. Then Z lies on an elliptic normal curve R of degree r in $\mathbf{P}^{r-1}$. Further, R is cut out by all quadrics containing Z if $r \geq 4$, and it is a plane cubic if $r = 3$.

3.6 We consider the case $h_{Z_0}(2) = 2r$. Since Z_0 lies on an elliptic normal curve by Lemma 3.5, we get $h_{Z_0}(3) = 3r - 1$. Then, it follows from (4) that

$$6r = h^0(C, \mathcal{O}(3K|_C)) \geq h_{C_0}(3) \geq h_{C_0}(2) + h_{Z_0}(3) = 6r.$$

Thus $h_{C_0}(3) = 6r$. Similarly, we can show $\delta h_{C_0}(n) = h_{Z_0}(n)$ for any $n \in \mathbf{N}$. Thus C_0 is projectively normal.

We turn our attention to S_0. By (2) and the pluri-genus formula, we have

$$4r + 3 - q(S) = h^0(S, \mathcal{O}(2K)) \geq h_{S_0}(2) \geq h_{S_0}(1) + h_{C_0}(2) = 4r + 3.$$

Thus $q(S) = 0$, $h^0(2K) = h_{S_0}(2)$ and $\delta h_{S_0}(2) = h_{C_0}(2)$. Quite similarly, we get $h^0(nK) = h_{S_0}(n)$ and $\delta h_{S_0}(n) = h_{C_0}(n)$ for any $n > 0$. Thus S_0 is also projectively normal, and we see that the multiplication map $Sym^n H^0(S, \mathcal{O}(K)) \to H^0(S, \mathcal{O}(nK))$ is surjective for any $n \geq 0$. Thus the canonical ring of S is generated in degree 1. This implies that S_0 is isomorphic to the canonical model. In particular, it has only RDP as its singularity.

We next show that S_0 is contained in an irreducible threefold W_0 of Δ-genus 1. We first assume that $r \geq 4$. By Lemma 3.5, we have $h^0(\mathcal{I}_{Z_0}(2)) = h^0(\mathcal{I}_R(2)) = r(r-3)/2$. Since S_0 is projectively normal, the linear system $|\mathcal{I}_{S_0}(2)|$ is restricted to $|\mathcal{I}_{Z_0}(2)|$ isomorphically. Thus $W_0 = Bs|\mathcal{I}_{S_0}(2)|$ is an irreducible threefold of Δ-genus 1, because $R = Bs|\mathcal{I}_{Z_0}(2)|$ is an elliptic normal curve of degree r. If $r = 3$, then Z_0 is contained in a plane cubic. Since S_0 is projectively normal, it is contained in a hypercubic W_0. Thus we get 2) of Theorem 3.1.

3.7 We next consider the case $h_{Z_0}(2) = 2r - 1$. We recall that C_0 and S_0 are linearly normal. Thus the linear system $|\mathcal{I}_{S_0}(2)|$ is restricted to $|\mathcal{I}_{Z_0}(2)|$ isomorphically. If we put $W_0 = Bs|\mathcal{I}_{S_0}(2)|$, then it is a threefold of Δ-genus 0, because $R = Bs|\mathcal{I}_{Z_0}(2)|$ is a rational normal curve of degree $r - 1$ by Lemma 3.4.

We show $q(S) = 0$. Since $h^0(\mathcal{I}_{Z_0}(2)) = h^0(\mathcal{I}_R(2)) = (r-1)(r-2)/2$, we get $h_{S_0}(2) = 4r + 2$. Since $h_{S_0}(2) \leq h^0(S, \mathcal{O}(2K)) = 4r + 3 - q(S)$, we have either $q(S) = 0$ or $q(S) = 1$. If $q(S) = 1$, then the Albanese map gives S a pencil of hyperelliptic curves of genus ≤ 3 by a result of Horikawa [11, V]. This is impossible, because the canonical map of S is birational. Thus $q(S) = 0$.

According to the classification of varieties of Δ-genus zero ([4], [8]), W_0 is either

A) $\mathbf{P}^3$, $(p_g = 4)$,

B) a hyperquadric $\subset \mathbf{P}^4$, $(p_g = 5)$,

C) a cone over $\mathbf{P}^2$ embedded into $\mathbf{P}^5$ by $|\mathcal{O}(2)|$, $(p_g = 7)$, or

D) a rational normal scroll $\subset \mathbf{P}^{p_g - 1}$, that is, the image of the total space of the $\mathbf{P}^2$-bundle

$$\pi : \mathbf{P}_{a,b,c} = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(b) \oplus \mathcal{O}_{\mathbf{P}^1}(c)) \to \mathbf{P}^1$$

under the holomorphic map induced by $|T|$, where T is the tautological divisor and a, b, c are integers satisfying

$$0 \leq a \leq b \leq c, \quad a + b + c = p_g - 3.$$

Thus there are the following three subcases:

D.1) $a > 0$: $W_0 \simeq \mathbf{P}_{a,b,c}$.

D.2) $a = 0$, $b > 0$: W_0 is a cone over the Hirzebruch surface Σ_{c-b} embedded into $\mathbf{P}^{p_g-2}$ by $|\Delta_0 + c\Gamma|$.

D.3) $a = b = 0$: W_0 is a generalized cone over the rational normal curve in $\mathbf{P}^{p_g-3}$. The ridge of W_0 is a line (see [7, §1] for the terminology).

We need not consider A) by the assumption $p_g \geq 5$. We claim that W_0 is singular in the case B). In fact, if W_0 is nonsingular hyperquadric, then S_0 is obtained as a hypersurface section of W_0. Then $\deg S_0$ must be even. This contradicts $\deg S_0 = 9$. Thus W_0 is singular. Then, since $\mathrm{rank}(W_0) \leq 4$, it can be represented as a rational normal scroll. We next exclude the case C). If C) is the case, then C_0 lies on the Veronese surface. Thus $\deg C_0$ must be even, contradicting $\deg C_0 = 15$. In summary, we get 1) of Theorem 3.1.

Definition 3.8 Let S and W_0 be as in Theorem 3.1. We say that S is of type I-0 or of type I-1 according to whether W_0 is of Δ-genus 0 or 1.

4 Surfaces of type I-1.

In this section, we let S be of type I-1 and W_0 the threefold of Δ-genus 1 on which the canonical image S_0 lies. We sometimes use the terminology in [6] and [7]. In particular, see [6, (5.6)] for the definition of a *Del Pezzo variety.*

4.1 By [7], if $W_0 \subset \mathbf{P}^{p_g-1}$ is an irreducible nondegenerate threefold of degree $p_g - 2$, then it is either

1) a hypercubic ($p_g = 5$),

2) a complete intersection of two hyperquadrics ($p_g = 6$),

3) a cone over a surface $V \subset \mathbf{P}^{p_g-2}$ of Δ-genus 1, where V is either

3a) the Veronese embedding into $\mathbf{P}^8$ of a quadric in $\mathbf{P}^3$ ($p_g = 10$),

3b) the image of $\mathbf{P}^2$ by the rational map associated with the linear system $|3l - \sum_{i=1}^{k} x_i|$, where l is a line on $\mathbf{P}^2$ and the x_i are points on $\mathbf{P}^2$ which are possibly infinitely near ($p_g = 11 - k$, $0 \leq k \leq 6$),

3c) a cone over a nonsingular elliptic curve, or

3d) a projection of a surface of Δ-genus 0 in $\mathbf{P}^{p_g-1}$ from a point,

4) a non-conic normal Del Pezzo threefold ($7 \le p_g \le 10$), or

5) a projection of a threefold of Δ-genus 0 in $\mathbf{P}^{p_g}$ from a point.

Not all varieties listed above can be W_0 in Theorem 3.1. The cases 3d) and 5) are excluded, because the canonical image S_0 and its general hyperplane section C_0 are both projectively normal. The case 3c) is also excluded because S is regular: In fact, if it is the case, we can easily show that S has an irrational pencil parametrized by an elliptic curve (cf. 4.3 below).

Thus we have $p_g(S) \le 11$ for any surface S of type I-1, and our W_0 is a normal Del Pezzo variety if $p_g \ge 7$. In particular, it is projectively normal (cf. [6, §5]).

Theorem 4.2 *Every surface S of type* I-1 *satisfies $5 \le p_g(S) \le 11$. The canonical image S_0 is a complete intersection of two hypercubics if $p_g = 5$, and it is a complete intersection of two hyperquadrics and a hypercubic if $p_g = 6$. If $p_g \ge 7$, then S_0 is a hypercubic section of a normal Del Pezzo threefold W_0.*

Proof. We use the notation of 3.6. Since W_0 is projectively normal, we can assume that R is nonsingular. Thus $h^0(\mathcal{I}_R(3)) = r(r+1)(r+2)/6 - 3r$. On the other hand, we have $h^0(\mathcal{I}_{Z_0}(3)) = r(r+1)(r+2)/6 - 3r + 1$. Thus there exists a hypercubic not containing R but Z_0. Since $\deg R = r$ and $\deg Z_0 = 3r$, we see that Z_0 is a hypercubic section of R. This gives Theorem 4.2, because W_0 and S_0 are both projectively normal.

q.e.d.

In the following, we examine the existence of surfaces of type I-1 in the cases 3) and 4) of 4.1.

4.3 Here we consider the case 3) of 4.1. Let v be the vertex of W_0. We denote by Λ_0 the pull-back to S by Φ_K of the linear system of hyperplanes through v. We let G be the fixed part of Λ_0 and put $\Lambda = \Lambda_0 - G$. Then Λ defines a rational map $\mu : S \to V$ and we have $K \sim H + G$, $H \in \Lambda$. We note that $KG = 0$, since $|K|$ has no base point. Then $3p_g - 6 = K^2 = KH = H^2 + HG \ge H^2 \ge (\deg \mu)(\deg V) = (\deg \mu)(p_g - 2)$. Thus $\deg \mu \le 3$. Since S is of type I, $\deg \mu \le 2$ leads us to a contradiction. Thus $\deg \mu = 3$, $HG = 0$ and $H^2 = 3p_g - 6$. Since $G^2 = 0$, we get $G = 0$ by Hodge's index theorem. Further, we see that μ is holomorphic. In fact, if Λ has a base point P, blowing S up at P and considering the strict transform $\tilde{\Lambda}$ of Λ, we would have $\tilde{H}^2 < H^2$ for $\tilde{H} \in \tilde{\Lambda}$ and conclude that the map induced by $\tilde{\Lambda}$ is of degree less than three.

The case 3a): As we saw above, a subsystem of $|K|$ induces a holomorphic map of degree 3 onto Σ_0 or a quadratic cone in $\mathbf{P}^3$. In the latter case, we can lift it to a holomorphic map $S \to \Sigma_2$ as in [10, p. 46]. Thus, in either case, we obtain a holomorphic map $h : S \to \Sigma_d$, with $d = 0$ or 2, satisfying $K \sim h^*(2\Delta_0 + (d+2)\Gamma)$. Let $W \to \Sigma_d$ be the line bundle associated with $H_0 = 2\Delta_0 + (d+2)\Gamma$ and, let w be the fiber coordinate of W. We show that S is birationally equivalent to a surface $S' \subset W$ defined by the equation

$$w^3 + \alpha_1 w^2 + \alpha_2 w + \alpha_3 = 0, \tag{5}$$

where $\alpha_i \in H^0(\Sigma_d, \mathcal{O}(iH_0))$. Since $p_g(S) = 10$ and $h^0(\Sigma_d, \mathcal{O}(H_0)) = 9$, there exists $\psi \in H^0(K)$ which is not induced by a section of $\mathcal{O}(H_0)$. Thus we get a holomorphic map $f : S \to W$ over h by putting $w = \psi$. Since h is of degree 3, f is birational onto $S' = f(S)$ by the choice of ψ. In $H^0(3K)$, we have the following elements:

$$\begin{array}{ll} \psi^3, & \\ \psi^2\alpha & \text{with } \alpha \in H^0(\Sigma_d, \mathcal{O}(H_0)), \\ \psi\beta & \text{with } \beta \in H^0(\Sigma_d, \mathcal{O}(2H_0)), \\ \gamma & \text{with } \gamma \in H^0(\Sigma_d, \mathcal{O}(3H_0)). \end{array} \tag{6}$$

These represents 84 sections. On the other hand, we have $h^0(3K) = 83$. Thus we can find a non-trivial relation of the form

$$\delta\psi^3 + \alpha\psi^2 + \beta\psi + \gamma = 0,$$

where δ is a constant and α, β, γ are as above. Since h is of degree 3, we see that δ is non-zero and there is no further relation among the sections in (6). Thus S' is defined by the equation of the form (5). Conversely, if we choose the α_i in (5) generic, then the obtained surface is nonsingular, minimal and satisfies $p_g = 10$, $q = 0$ and $c_1^2 = 24$.

The case 3b): Let $\tilde{V}$ be the projective plane blown up at $k = 11 - p_g$ points $x_1, \cdots, x_k$ and let $\lambda : \tilde{V} \to \mathbf{P}^2$ be the natural map. Let $H_0 = 3\lambda^* l - \sum \lambda^{-1}(x_i)$ be the pull-back of the hyperplane of $\mathbf{P}^{p_g-2}$. We denote by W the line bundle associated with H_0, and let w be its fiber coordinate. We show that S is birationally equivalent to a hypersurface S' of W defined by

$$w^3 + \alpha_1 w^2 + \alpha_2 w + \alpha_3 = 0, \tag{7}$$

where $\alpha_i \in H^0(\tilde{V}, \mathcal{O}(iH_0))$, $1 \leq i \leq 3$.

Since $\mathcal{O}(-H_0)$ is the canonical sheaf of $\tilde{V}$, we have

$$h^0(H_0) = p_g - 1, \;\; h^0(2H_0) = 3p_g - 5, \;\; h^0(3H_0) = 6p_g - 11,$$

by the Riemann-Roch theorem and the Kodaira vanishing theorem. Let $\phi_0, \cdots, \phi_{p_g-2}, \psi$ be a basis of $H^0(S, \mathcal{O}(K))$ such that $\phi_0, \cdots, \phi_{p_g-2}$ span the module of Λ. Since $h^0(3K) = 10p_g - 27$, the following $10p_g - 26$ products

$$\psi^3, \quad \phi_i\psi^2, \quad \phi_i\phi_j\psi, \quad \phi_i\phi_j\phi_k$$

in $H^0(3K)$ are linearly dependent. Thus S is birationally equivalent to a triple covering of V defined by

$$u^3 + \alpha(z)u^2 + \beta(z)u + \gamma(z) = 0$$

where α, β and γ are homogeneous forms of respective degree 1, 2 and 3 in the homogeneous coordinates $(z_0 : \cdots : z_{p_g-2})$ of $\mathbf{P}^{p_g-2}$ and $\psi = \Phi_K^* u$. This triple covering induces via $\tilde{V} \to V$ a triple covering S' of $\tilde{V}$. Then the equation of S' is of the form (7). Conversely, if we choose the α_i in (7) generic, then the obtained surface is nonsingular and satisfies $c_1^2 = 3p_g - 6$.

4.4 Here we consider the case 4) of 4.1 assuming that W_0 is nonsingular. In this case, we can determine the divisor class of S_0 by Theorem 4.2 once we know the structure of W_0. For the classification of polarized manifolds of Δ-genus one, see Fujita [5].

i) $p_g = 7$: Let $Gr(2, 5)$ be the Grassmannian of two planes in $\mathbf{C}^5$ embedded into $\mathbf{P}^9$ by the Plücker embedding. Then W_0 is obtained by cutting $Gr(2, 5)$ three times by hyperplanes. S_0 is a hypercubic section of W_0.

ii) $p_g = 8$:

ii-1) $W_0 = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ embedded by $|H_1 + H_2 + H_3|$, where H_i is the pull-bak of a point of the i-th factor. $S_0 \sim 3(H_1 + H_2 + H_3)$.

ii-2) $W_0 = \mathbf{P}(\Theta_{\mathbf{P}^2})$, where $\Theta_{\mathbf{P}^2}$ is the tangent sheaf of $\mathbf{P}^2$. If H denotes the tautological divisor, then W_0 is embedded by $|H|$. $S_0 \sim 3H$.

iii) $p_g = 9$: W_0 is the blowing-up of $\mathbf{P}^3$ at one point. If we denote by H and E the pull-back of a plane of $\mathbf{P}^3$ and the exceptional divisor, respectively, then W_0 is embedded by $|2H - E|$. $S_0 \sim 6H - 3E$.

iv) $p_g = 10$: $W_0 = \mathbf{P}^3$ embedded by $|\mathcal{O}(2)|$ and S_0 is a sixtic surface in $\mathbf{P}^3$.

It is clear that S_0 with only RDP exists in each case.

4.5 We consider the case 4) of 4.1 assuming that W_0 is singular. From [7], we know that this case occurs only when $p_g = 7$, 8, and that W_0 is represented as the image of the following W:

i) $p_g = 7$:

i-1) Consider the $\mathbf{P}^2$-bundle $\tilde{\pi} : \mathbf{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(\Delta_0 + 2\Gamma)) \to \Sigma_1$. If we denote by $\tilde{T}$ the tautological divisor, then $W \sim \tilde{T} + \tilde{\pi}^*(\Delta_0 + \Gamma)$.

i-2) Consider the $\mathbf{P}^1$-bundle $\tilde{\pi} : \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(T)) \to \mathbf{P}_{1,1,1}$ (see 3.7 for the notation). If $\tilde{T}$ denotes the tautological divisor, then $W \sim \tilde{T} + \tilde{\pi}^*(T - F)$, where F is a fiber of $\mathbf{P}_{1,1,1} \to \mathbf{P}^1$.

ii) $p_g = 8$:

ii-1) Consider the $\mathbf{P}^2$-bundle $\mathbf{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2))$ over $\mathbf{P}^2$. If we denote by $\tilde{T}$ and $\tilde{F}$ the tautological divisor and the pull-back of a line in $\mathbf{P}^2$, respectively, then $W \sim \tilde{T} + \tilde{F}$.

ii-2) Consider the $\mathbf{P}^2$-bundle $\tilde{\pi} : \mathbf{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(\Delta_0 + 3\Gamma)) \to \Sigma_2$. If $\tilde{T}$ denotes the tautological divisor, then W is a nonsingular member of $|\tilde{T} + \tilde{\pi}^*(\Delta_0 + \Gamma)|$.

ii-3) Consider the $\mathbf{P}^1$-bundle $\tilde{\pi} : \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(T)) \to \mathbf{P}_{1,1,2}$. If $\tilde{T}$ denotes the tautological divisor, then $W \sim \tilde{T} + \tilde{\pi}^*(T - 2F)$.

ii-4) Consider the $\mathbf{P}^2$-bundle $\tilde{\pi} : \mathbf{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(\Delta_0 + 2\Gamma)) \to \Sigma_0$. If $\tilde{T}$ denotes the tautological divisor, then W is a nonsingular member of $|\tilde{T} + \tilde{\pi}^*\Delta_0|$.

In each case, the natural map $W \to W_0$ is induced by $|\tilde{T}|$. We have $H^0(W, \mathcal{O}(k\tilde{T})) \simeq H^0(W_0, \mathcal{O}(k))$ for any $k > 0$, and W has some mild singularities in general (see [7]). One can easily check the existence of a surface $S' \in |3\tilde{T}|_W|$ with only RDP satisfying $\omega_{S'}^2 = 3h^0(\omega_{S'}) - 6$.

5 Lifting of the canonical maps.

From now on, we study surfaces of type I-0. Let S be of type I-0 and W_0 the threefold of Δ-genus zero on which the canonical image S_0 lies. Then W_0 is a rational normal scroll as we saw in §3. In this section, we assume that it is singular, and discuss whether we can lift the canonical map to a holomorphic map into a nonsingular model of W_0.

5.1 We consider the case D.2) of 3.7. That is, W_0 is a cone over the Hirzebruch surface Σ_{c-b} embedded into $\mathbf{P}^{p_g-2}$ by $|\Delta_0 + c\Gamma|$, where $0 < b \leq c$ and $p_g = b + c + 3$. We denote by $\hat{W}$ the total space of the $\mathbf{P}^1$-bundle $\varpi : \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(\Delta_0 + c\Gamma)) \to \Sigma_{c-b}$. Then it is a nonsingular model of W_0. We let H be the tautological divisor of $\hat{W}$ and denote by H_∞ the unique divisor with $H_\infty \sim H - \varpi^*(\Delta_0 + c\Gamma)$. Then the natural map $\hat{W} \to W_0$ is induced by $|H|$, and H_∞ is contracted to the vertex of W_0.

Let Λ_0 be the pull-back to S by Φ_K of the linear system of hyperplanes through the vertex of W_0. We let G be the fixed part of Λ_0 and put $\Lambda = \Lambda_0 - G$. We have $KG = 0$ since $|K|$ is free from base points. The linear system Λ induces a surjective rational map $\mu : S \to \Sigma_{c-b}$. We let $\sigma : \hat{S} \to S$ be the composition of quadric transformations which is the shortest among those with the property that the variable part of $\sigma^*\Lambda$ is free from base points. We denote by $\hat{K}$ the canonical divisor of $\hat{S}$. Then $\hat{K} \sim \sigma^*K + E$, where E is the exceptional divisor of σ. We have a holomorphic map $\hat{\mu} : \hat{S} \to \Sigma_{c-b}$ and $\sigma^*K \sim M + \hat{E} + \sigma^*G$, where $M = \hat{\mu}^*(\Delta_0 + c\Gamma)$ and $\hat{E}$ is a sum of exceptional curves of σ satisfying $\mathrm{Supp}(\hat{E}) = \mathrm{Supp}(E)$ and $\hat{E} \geq E$. We remark that $\deg \hat{\mu}$ is greater than two. Since

$$3(b+c+1) = (\sigma^*K)^2 = (\sigma^*K)M \geq M^2 = (\deg \hat{\mu})(b+c),$$

we have either

1) $\deg \hat{\mu} = 3$, or

2) $\deg \hat{\mu} = 4$ and $b+c = 2,\ 3$.

We also remark that $(\sigma^*K)\hat{E} = M\hat{E} + \hat{E}^2 = 0$ and $(\sigma^*K)(\sigma^*G) = M(\sigma^*G) + G^2 = 0$.

5.2 We consider the case 1) of 5.1. We have $M(\hat{E} + \sigma^*G) = 3$. Since $KG = 0$, G^2 is even and so is $M(\sigma^*G)$. Thus we have either

i) $M\hat{E} = 1,\ M(\sigma^*G) = 2$, or

ii) $M\hat{E} = 3,\ M(\sigma^*G) = 0$.

We first consider the case i). Then $\hat{E}^2 = -1$, $G^2 = -2$. Let E_0 be a (-1)-curve on $\hat{S}$. Then it is a common irreducible component of $\hat{E}$ and E. Further, we have $ME_0 > 0$, because, if $ME_0 = 0$, we can contract E_0 contradicting that σ is the shortest. Thus $M\hat{E} = ME = ME_0 = 1$. Then, similarly as in [10, §1], we can show that $\hat{E} = E$ is a (-1)-curve E_0, which we represent by E for the sake of simplicity. Thus, σ is the blowing-up at $\sigma(E)$. Since $ME = 1$, we see that $\hat{\mu}$ maps E biholomorphically onto a nonsingular rational curve. Put $\hat{\mu}(E) \sim \alpha\Delta_0 + \beta\Gamma$, where α and β are nonnegative integers. Since $(\Delta_0 + c\Gamma)\hat{\mu}(E) = 1$, we get $(\alpha,\ \beta) = (1,\ 0)$ or $(0,\ 1)$. We remark that $(\alpha,\ \beta) = (1,\ 0)$ occurs only when $b = 1$, and that, if $b = c = 1$, we can assume $(\alpha,\ \beta) = (0,\ 1)$ by considering another ruling of Σ_0.

We have $\hat{K} \sim \hat{\mu}^*(\Delta_0 + c\Gamma) + 2E + \sigma^*G$. Since $(\hat{\mu}^*\Gamma)^2 = 0$, $\hat{K}(\hat{\mu}^*\Gamma) = 3 + 2E(\hat{\mu}^*\Gamma) + (\sigma^*G)(\hat{\mu}^*\Gamma)$ is an even integer. Thus $(\sigma^*G)(\hat{\mu}^*\Gamma)$ is a positive odd integer. Since $KG = 0$, G consists of (-2)-curves. Thus any irreducible component of σ^*G is a nonsingular rational curve as well. Since $M(\sigma^*G) = 2$, we have at most two irreducible components of σ^*G having positive intersection number with M. If we have two such components G_0 and G_1 with $MG_0 = MG_1 = 1$, then $\hat{\mu}(\sigma^*G - G_0 - G_1)$ cannot be a divisor, because M is the pull-back of an ample divisor and $M(\sigma^*G - G_0 - G_1) = 0$. Thus $(\sigma^*G)\hat{\mu}^*\Gamma = (G_0 + G_1)\hat{\mu}^*\Gamma$ is odd. From this and $MG_i = 1$, we see that one of them, say G_0, is mapped to Δ_0 and G_1 is mapped to a fiber Γ. In particular, we get $b = 1$. If we have a component G_0 with $MG_0 = 2$, then $\hat{\mu}(\sigma^*G - G_0)$ is not a divisor and we have either $\hat{\mu}(G_0) = \Delta_0$, $b = 2$ or $\hat{\mu}(G_0) \sim \Delta_0 + \Gamma$, $b = 1$ by a simple calculation. In the last case, we have $c \leq 2$ because the irreducibility of G_0 implies $(\Delta_0 + \Gamma)\Delta_0 = 2 - c \geq 0$. In either case, we have $(\sigma^*G)(\hat{\mu}^*\Gamma) = 1$.

As a consequence, $|\hat{\mu}^*\Gamma|$ is a pencil of curves of genus 3 (if $\hat{\mu}(E) = \Gamma$) or 4 (if $\hat{\mu}(E) = \Delta_0$). Put $D = \sigma_*\hat{\mu}^*\Gamma$. In the former case, $|D|$ is a pencil of curves of genus 3 without base points. As we shall see in Lemma 5.3, there is a lifting $f : S \to \mathbf{P}_{0,b,c}$ of the canonical map. In the latter case, $|D|$ is a pencil of curves of genus 4 with one base point. We show that there is a natural map $f : \hat{S} \to \mathbf{P}_{0,b,c}$ over the holomorphic map induced by $|\sigma^*K|$. Let $\xi \in H^0(G)$ and $e \in H^0(\hat{S}, \mathcal{O}(E))$ be sections satisfying $(\xi) = G$ and $(e) = E$. Then we can find a section $x_0 \in H^0(K)$ such that $\mathrm{Supp}((\sigma^*x_0))$ does not intersect with $\sigma^*G + E$. Then the pair $(\sigma^*x_0, e\sigma^*\xi)$ defines a holomorphic map $\hat{f} : \hat{S} \to \hat{W}$ such that $\hat{f}^*H_\infty = \sigma^*G + E$. In particular, we have $\sigma^*K \sim M + \hat{f}^*H_\infty \sim \hat{f}^*H$. We remark that there is a holomorphic map $\nu : \hat{W} \to \mathbf{P}_{0,b,c}$ which contracts the divisor H_∞ to a nonsingular rational curve and satisfies $\nu^*T \sim H$. Thus we get the desired map by putting $f = \nu \circ \hat{f}$.

We next consider the case ii). By Hodge's index theorem, we have $G = 0$. Since $\hat{K}M + M^2 = 6(b+c) + 3 + ME$ is even, ME is odd. Then $ME = 1$ or 3, since $M\hat{E} = 3$ and $\hat{E} \geq E$. If $ME = 3$, then we have $M(\hat{E} - E) = 0$. Thus $\hat{\mu}(\hat{E} - E)$ is at most 0-dimensional. But then, $\hat{K}(\hat{\mu}^*\Gamma) = 3 + 2E(\hat{\mu}^*\Gamma)$ is odd, a contradiction. If $ME = 1$, then we can find a (-1)-curve E_0 with $ME_0 = 1$. No irreducible component of $E_1 := E - E_0$ is a (-1)-curve, because $ME_1 = 0$. Thus $\hat{E} = 3E_0 + \hat{E}_1$ with $M\hat{E}_1 = 0$. We have $\hat{K} \sim M + 4E_0 + E_1 + \hat{E}_1$. Since $\hat{K}E_0 = -1$, we have $(E_1 + \hat{E}_1)E_0 = 2$. On the other hand, we have $-3 = \hat{E}^2 = (3E_0 + \hat{E}_1)^2 = -9 + 6E_0\hat{E}_1 + (\hat{E}_1)^2$. Thus $(\hat{E}_1)^2 = 6 - 6E_0\hat{E}_1$. Since $M\hat{E}_1 = 0$, we get $(\hat{E}_1)^2 \leq 0$ by Hodge's index theorem. Thus $E_0\hat{E}_1 \geq 1$. If $E_0\hat{E}_1 = 2$, then $E_0E_1 = 0$. This means $E_1 = 0$, because, otherwise, E_1 must contain a (-1)-curve. Then $\hat{E}_1$ is also zero, a contradiction. If $E_0\hat{E}_1 = 1$, then $(\hat{E}_1)^2 = 0$. Since $M\hat{E}_1 = 0$, we have $\hat{E}_1 = 0$ by Hodge's index theorem, a contradiction. In summary, the case ii) is excluded.

Lemma 5.3 *Let the situation be in the case i) of* 5.2 *and suppose that S has a pencil $|D|$ of curves of genus* 3 *described there. Then the canonical map can be lifted to a holomorphic map $f : S \to \mathbf{P}_{0,b,c}$.*

Proof. We choose a general $D \in |D|$ and put $\hat{D} = \sigma^*D$. Then $\hat{D} \in |\hat{\mu}^*\Gamma|$. Since $E\hat{D} = 0$, $\sigma^*K|_{\hat{D}}$ is the canonical divisor $K_{\hat{D}}$ of $\hat{D}$. We consider the commutative diagram

$$\begin{array}{ccccc}
0 \to H^0(\sigma^*K - (i+1)\hat{D}) \to & & H^0(\sigma^*K - i\hat{D}) \to & & H^0(\hat{D}, K_{\hat{D}}) \\
\uparrow & & \uparrow & & \uparrow \\
0 \to H^0(\Delta_0 + (c-i-1)\Gamma) \to & & H^0(\Delta_0 + (c-i)\Gamma) \to & & H^0(\Gamma, \mathcal{O}(1))
\end{array}$$

for $i \geq 0$. We remark that the vertical maps are injective, since $\sigma^*K \sim \mu^*(\Delta_0 + c\Gamma) + E + \sigma^*G$.

We measure $m(i) := \dim_{\mathbf{C}} \mathrm{Im}\{H^0(\sigma^*K - i\hat{D}) \to H^0(K_{\hat{D}})\}$. Since Φ_K is birational, the natural map $H^0(\sigma^*K) \to H^0(K_{\hat{D}})$ is surjective. Thus $h^0(\sigma^*K - \hat{D}) = p_g - 3 = h^0(\Delta_0 + (c-1)\Gamma)$. Then, by a diagram chasing, we see that $H^0(\Delta_0 + (c-i)\Gamma) \to$

$H^0(\sigma^*K - i\hat{D})$ is bijective for $i > 0$. Thus 0 is the greatest integer such that $m(i) = 3$. Similarly, b (resp. c) is the greatest integer such that $m(i) = 2$ (resp. $m(i) = 1$) if $b < c$, and b is the greatest integer such that $m(i) > 0$ if $b = c$.

Since $H^0(K - iD) \simeq H^0(\sigma^*K - i\hat{D})$ for any i, we can choose three sections $x_0 \in H^0(K)$, $x_1 \in H^0(K-bD)$ and $x_2 \in H^0(K-cD)$ such that they span $H^0(K_D)$. Then the triple (x_0, x_1, x_2) defines a rational map $f : S \to \mathbf{P}_{0,b,c}$. We can assume it holomorphic, since $|K|$ has no base point. *q.e.d.*

5.4 Suppose we are in the case 2) of 5.1. We have $M^2 = 4(b+c)$ and $M(\hat{E} + \sigma^*G) = 3 - (b+c)$, where $(b, c) = (1, 1)$ or $(1, 2)$.

We first consider the case $(b, c) = (1, 1)$. If $M\hat{E} = 0$ and $M(\sigma^*G) = 1$, then we get $G^2 = -1$. This contradicts that $KG + G^2$ is even. Thus we have $M\hat{E} = 1$, $M(\sigma^*G) = 0$. Then $G^2 = 0$. Thus $G = 0$ by Hodge's index theorem. Further, we can show that $\hat{E} = E$ is a (-1)-curve and $\hat{\mu}(E) = \Delta_0$ or a fiber Γ as in 5.2. We can assume that $\hat{\mu}(E)$ is a fiber by considering another ruling of Σ_0 if $\hat{\mu}(E) = \Delta_0$. Then $|\hat{\mu}^*\Gamma|$ is a pencil of curves of genus 3. Since $E\hat{\mu}^*\Gamma = 0$, we see that S also has a pencil $|D|$, $D = \sigma_*\hat{\mu}^*\Gamma$, of curves of genus 3 without base points.

We next consider the case $(b, c) = (1, 2)$. Then we have $\hat{E} = G = 0$ by Hodge's index theorem. Thus σ is the identity map and $K \sim \mu^*(\Delta_0 + 2\Gamma)$. This shows that S has a pencil $|D|$, $D = \mu^*\Gamma$, of curves of genus 3.

In either case, we can show that the canonical map can be lifted to a map $f : S \to \mathbf{P}_{0,b,c}$ as in Lemma 5.3.

5.5 We consider the case D.3) of 3.7. Thus W_0 is a generalized cone over a rational normal curve. By the argument similar to [11, I, Lemma 1], we have $K \sim cD + G$, where $|D|$ is a pencil and G is a (possibly zero) effective divisor corresponding to the ridge of W_0. Since $K^2 = 3c + 3 \geq cKD$, we have $KD \leq 4$. By Hodge's index theorem and the fact that $KD + D^2$ is even, we are left the following possibilities:

i) $KD = 3$, $D^2 = 1$, $DG = 1$ $\quad (c = 2)$,

ii) $KD = 4$, $D^2 = 0$, $DG = 4$ $\quad (c = 2, 3)$.

We show that i) is impossible. We remark that a general $D \in |D|$ is nonsingular, since $D^2 = 1$. Since S is of type I, Φ_K maps D birationally onto its image. However, since $KD = 3$, we have $h^0(D, \mathcal{O}(K|_D)) \leq 2$ by Clifford's theorem. Thus $\Phi_K(D)$ is (at most) a rational curve, a contradiction.

In the case ii), we let ξ be the section of $\mathcal{O}(G)$ with $(\xi) = G$ and let x_0, x_1 be two independent element in $H^0(K)$ which correspond to hyperplanes not containing the rigde of W_0. We can assume that $\mathrm{Supp}((x_0)) \cap \mathrm{Supp}((x_1)) \cap G = \emptyset$ since $|K|$ has no base points. Then the triple (x_0, x_1, ξ) defines the holomorphic map $f : S \to \mathbf{P}_{0,0,c}$.

We summarize the above results in the following:

Proposition 5.6 *Let S be a surface of type* I-0 *whose canonical image is contained in a singular rational normal scroll. Then it has a pencil $|D|$ of nonhyperelliptic curves of genus* 3 *or* 4.

1) *If $g(D) = 3$, then $|D|$ has no base point and there is a lifting $f : S \to \mathbf{P}_{0,b,c}$ of the canonical map. In this case, $0 \le b \le 2$ and, if $b = 0$, then $c = 2, 3$.*

2) *If $g(D) = 4$, then $|D|$ has a base point P. Let $\sigma : \hat{S} \to S$ be the blowing-up with center P. Then there is a natural map $f : \hat{S} \to \mathbf{P}_{0,b,c}$ over the holomorphic map induced by $|\sigma^* K|$. In this case, $b = 1$.*

Further, it can be assumed that $g(D) = 3$ if $p_g = 5$.

6 Divisor classes.

We assume that $p_g \ge 5$ as usual.

6.1 Let S be of type I-0. If $a > 0$, we denote by $f : S \to \mathbf{P}_{a,b,c}$ the natural map induced by Φ_K. We let f have the same meaning as in Proposition 5.6, if $a = 0$. Put $S^* = f(S)$ (or $f(\hat{S})$ in case 2), Proposition 5.6). We choose a general member $C \in |K|$ and put $C^* = f(C)$ (or $f(\sigma^* C)$ in case 2), Proposition 5.6).

The Picard group of $W := \mathbf{P}_{a,b,c}$ is generated by the tautological divisor T and a fiber F. In particular, we have $K_W \sim -3T + (p_g - 5)F$, since $a + b + c = p_g - 3$. We have $T^3 = (p_g - 3)T^2 F$ in the Chow ring of W.

We determine the linear equivalence class of S^*. Put $S^* \sim \alpha T + \beta F$ and denote by $g : S^* \to \mathbf{P}^1$ the holomorphic map induced by the projection map of W. Then every fiber of g is a plane curve of degree α. Thus we get $\alpha \ge 4$, since, otherwise, S^* has a pencil of curves of genus less than 2 contradicting that it is birational to the surface S of general type. Since $\deg S_0 = 3p_g - 6$, we have

$$3p_g - 6 = T^2(\alpha T + \beta F) = \alpha(p_g - 3) + \beta$$

Thus $\beta = (3 - \alpha)(p_g - 3) + 3$. Since C is of genus $3p_g - 5$, the arithmetic genus of C^* can be written as $p_a(C^*) = 3p_g - 5 + \delta$ with some nonnegative integer δ. Since we have

$$\begin{aligned} p_a(C^*) &= \tfrac{1}{2}T((\alpha - 2)T + (\beta + p_g - 5)F)(\alpha T + \beta F) + 1 \\ &= \tfrac{1}{2}[3(\alpha - 2)(p_g - 2) + \alpha(\beta + p_g - 5)] + 1, \end{aligned}$$

we get $(\alpha-4)(\beta+1) = 2\delta - 4$. Recall that, if $p_g = 5$, we can assume $\alpha = 4$ by Proposition 5.6. So we get the following list:

(s.1) $S^* \sim 5T - 5F$ if $p_g = 7$, $p_a(C^*) = 16$.

(s.2) $S^* \sim 5T - 3F$ if $p_g = 6$, $p_a(C^*) = 14$,

(s.3) $S^* \sim 4T - (p_g - 6)F$ if $p_a(C^*) = 3p_g - 3$,

This and Theorem 4.2 in particular show the following:

Theorem 6.2 *Let S be a surface of type* I. *If $p_g(S) \geq 12$, then it has a pencil of nonhyperelliptic curves of genus* 3. *If S is of type* I-0, *then the same holds for $p_g(S) \geq 8$.*

Lemma 6.3 *The cases* (s.1) *and* (s.2) *cannot occur when $a > 0$.*

Proof. We assume $a > 0$ and show that this leads us to a contradiction.

We first consider the case (s.1). Since $p_g = 7$, we have $(a, b, c) = (1, 1, 2)$. We can identify S^* with the canonical image S_0. It has only isolated singular points, because $p_a(C^*) = g(C)$ implies that its general hyperplane section is nonsingular. Thus it is normal by Serre's criterion. By a direct calculation, we have $\chi(\mathcal{O}_{S^*}) = \chi(\mathcal{O}_W) - \chi(\mathcal{O}_W(-S^*)) = 4$. This is impossible, because S is the minimal resolution of S^* and $\chi(\mathcal{O}_S) = 8 > \chi(\mathcal{O}_{S^*})$.

We next consider the case (s.2). Since $p_g = 6$, we have $(a, b, c) = (1, 1, 1)$. Thus $W \simeq \mathbf{P}^1 \times \mathbf{P}^2$. Under this identification, T and F correspond to $H_1 + H_2$ and H_1, respectively, where H_i is the pull-back of a hyperplane in $\mathbf{P}^i$. Thus $S^* \sim 2H_1 + 5H_2$ on $\mathbf{P}^1 \times \mathbf{P}^2$. Then it is a double covering of $\mathbf{P}^2$ via the projection map $W \to \mathbf{P}^2$. This contradicts that it is birational to a type I surface. *q.e.d.*

Let the situation be as in 6.1. In the cases (s.2) and (s.3), the curve C^* is singular since $p_a(C^*) > g(C)$. In the rest of this section, we study its singularity.

6.4 We denote by $\phi : C \to C_0 \subset \mathbf{P}^r$, $r = p_g - 2$, the restriction of the canonical map of S to C. Then C_0 is contained in a surface scroll V. Recall that V is one of the surfaces in c) and d) of 2.1. If V is in d), then it is singular. In this case, however, there exists the lifting $C \to \Sigma_{r-1}$ of ϕ (see 5.5). Thus, in either case, we have a holomorphic map $f_C : C \to \Sigma_d$ over ϕ, where $r - d - 1$ is a nonnegative even integer and the natural map $\Sigma_d \to V \subset \mathbf{P}^r$ is induced by $|H_0|$, $H_0 = \Delta_0 + ((r-1+d)/2)\Gamma$. Then C^* is the image of f_C and Σ_d can be identified with a member of $|T|$. Further, $T|_{\Sigma_d} \sim H_0$ and $F|_{\Sigma_d} \sim \Gamma$. Since $2K|_C$ is the canonical divisor K_C of C, we get $K_C \sim 2f_C^* H_0$.

Since C^* is irreducible, we have $C^*\Delta_0 \geq 0$. Then we get the following list corresponding to (s.1), (s.2) and (s.3):

(c.1) $C^* \sim 5\Delta_0 + 10\Gamma$ on Σ_2, $(r = 5,\ p_a(C^*) = 16)$.

(c.1)$'$ $C^* \sim 5\Delta_0 + 5\Gamma$ on Σ_0, $(r = 5,\ p_a(C^*) = 16)$,

(c.2) $C^* \sim 5\Delta_0 + 7\Gamma$ on Σ_1, $(r = 4,\ p_a(C^*) = 14)$,

(c.3) $C^* \sim 4\Delta_0 + (r + 2 + 2d)\Gamma$ on Σ_d, $(p_a(C^*) = 3r + 3)$,

Lemma 6.5 *The case (c.1)$'$ of 6.4 can be excluded.*

Proof. Since $p_a(C^*) = g(C)$, we can identify C^* with C. By the adjunction formula and $K_C \sim 2H_0|_C$, we have

$$K_C \sim (3\Delta_0 + 3\Gamma)|_C \sim (2\Delta_0 + 4\Gamma)|_C.$$

Thus we get $\Delta_0|_C \sim \Gamma|_C$. This is impossible, because C meets a general member of $|\Delta_0|$ at distinct five points which are not on the same fiber Γ. *q.e.d.*

Lemma 6.6 *Assume that C^* is of type (c.2) in 6.4, then it has a singular point P of multiplicity 2 on Δ_0.*

Proof. Since $p_a(C^*) - g(C) = 1$, P is the unique singular point of multiplicity 2. Let $\lambda : V^* \to \Sigma_1$ be the blowing-up with center P and put $E = \lambda^{-1}(P)$. Then the proper transform of C^* is isomorphic to C and it is linearly equivalent to $\lambda^*(5\Delta_0 + 7\Gamma) - 2E$. Thus we have $K_C \sim (\lambda^*(3\Delta_0 + 4\Gamma) - E)|_C$ by the adjunction formula. On the other hand, we have $K_C \sim 2\lambda^* H_0|_C \sim \lambda^*(2\Delta_0 + 4\Gamma)|_C$. From these, it follows $(\lambda^*\Delta_0 - E)|_C = 0$. This means that P is on Δ_0. *q.e.d.*

Lemma 6.7 *Assume that C^* is of type (c.3) in 6.4. Then it has two singular points P_1, P_2 of multiplicity 2, which are possibly infinitely near. Further, they are on the same fiber of Σ_d.*

Proof. Since $p_a(C^*) - g(C) = 2$, the first assertion is clear. Assume that P_1 and P_2 are on the distinct fibers Γ_1 and Γ_2, respectively. We let $\lambda : V^* \to \Sigma_d$ be the blowing-up with center $P_1 \cup P_2$, and put $E_i = \lambda^{-1}(P_i)$, $1 \leq i \leq 2$. Since the proper transform of C^* is isomorphic to C, we have $C \sim \lambda^*(4\Delta_0 + (r+2+2d)\Gamma) - 2E_1 - 2E_2$ and $K_C \sim (\lambda^*(2\Delta_0 + (r+d)\Gamma) - E_1 - E_2)|_C$ by the adjunction formula. On the other hand, since $K_C \sim 2\lambda^* H_0|_C$, we have $K_C \sim \lambda^*(2\Delta_0 + (r-1+d)\Gamma)|_C$. Thus, if we denote the proper transform of Γ_i by $\widetilde{\Gamma}_i$, we get $\lambda^*\Gamma|_C \sim \widetilde{\Gamma}_1|_C + \widetilde{\Gamma}_2|_C$. This implies that $\widetilde{\Gamma}_1|_C$

and $\tilde{\Gamma}_2|_C$ are the pull-back of a point via the natural map $C \to \mathbf{P}^1$ defined by $|\lambda^*\Gamma|_C|$. This contradicts our initial assumption $\Gamma_1 \neq \Gamma_2$.

We next suppose that C^* has an infinitely near double point P_1. We denote by Γ_1 the fiber passing through P_1. Let $\lambda_1 : V_1 \to \Sigma_d$ be the blowing up at P_1 and denote by $\hat{E}_1$, $\hat{\Gamma}_1$ and $\hat{C}$ the exceptional curve, the proper transforms of Γ_1 and C^*, respectively. Then $\hat{C}$ still has a singular point P_2 of multiplicity 2 on $\hat{E}_1$. We must show that $\hat{\Gamma}_1$ contains P_2. Assume that this is not the case. Let $\lambda_2 : V^* \to V_1$ be the blowing-up of V_1 at P_2 and put $E_2 = \lambda_2^{-1}(P_2)$. We denote by E_1 and $\tilde{\Gamma}_1$ the proper transforms by λ_2 of $\hat{E}_1$ and $\hat{\Gamma}_1$, respectively. The proper transform of $\hat{C}$ by λ_2 can be identified with C. We put $\lambda = \lambda_2 \circ \lambda_1 : V^* \to \Sigma_d$. Since $C \sim \lambda^*C^* - 2E_1 - 4E_2$, it does not intersect with E_1. Then, using $K_C \sim 2\lambda^*H_0|_C$ and the adjunction formula, we get

$$\begin{aligned} K_C &\sim \lambda^*(2\Delta_0 + (r-1+d)\Gamma)|_C \\ &\sim \lambda^*(2\Delta_0 + (r+d)\Gamma - E_1 - 2E_2)|_C. \end{aligned}$$

Since $\tilde{\Gamma}_1 \sim \lambda^*\Gamma - E_1 - E_2$ and $CE_1 = 0$, we have $2\tilde{\Gamma}_1|_C \sim \lambda^*\Gamma|_C$. This is impossible by the same reasoning as in the previous case. Thus P_2 is contained in $\hat{\Gamma}_1$. *q.e.d.*

7 Surfaces of type I-0: The case (s.1).

By Lemma 6.3 and the definition of the map f, we see that the cases (s.1) and (s.2) of 6.1 correspond to 2) of Proposition 5.6. In this section, we prove the existence of surfaces of type (s.1).

7.1 Since $p_g = 7$, we have $(a, b, c) = (0, 1, 3)$ by Proposition 5.6 and Lemma 6.3. We choose sections X_0, X_1 and X_3 of $\mathcal{O}(T)$, $\mathcal{O}(T-F)$ and $\mathcal{O}(T-3F)$, respectively, such that they form a system of homogeneous fiber coordinates on each fiber of $W \to \mathbf{P}^1$. Then the equation of S^* can be written as

$$\sum_{i,j\geq 0, i+j\leq 5} \phi_{ij} X_0^{5-i-j} X_1^i X_2^j = 0,$$

where ϕ_{ij} is a homogeneous form of degree $i+3j-5$ on $\mathbf{P}^1$. Thus S^* is singular along the rational curve B defined by $X_1 = X_2 = 0$. Note that B is contracted to the vertex of W_0 under the holomorphic map induced by $|T|$. Since $p_a(C^*) = g(C)$, we see that S^* has only isolated singularity except for B.

We let $\nu : \hat{W} \to W$ be the blowing-up with center B. Since $\hat{W}$ is nothing but the manifold appeared in §5, we use the notation there. In particular, we have $\nu^*T \sim H$, $\nu^*F \sim \varpi^*\Gamma$ and $\nu^{-1}(G_0) = H_\infty$. Let S' be the proper transform of S^* by ν. Then we have $S' = \hat{f}(\hat{S})$, where $\hat{f} : \hat{S} \to \hat{W}$ is the holomorphic map defined in 5.2. If the multiplicity of S^* at a generic point of B is k, then S' is linearly equivalent to

$5H - 5\varpi^*\Gamma - kH_\infty$. From this, we infer readily $k = 2$, since $\hat{\mu} : \hat{S} \to \Sigma_2$ is of degree 3. Thus $S' \sim 3H + \varpi^*(2\Delta_0 + \Gamma)$ on $\hat{W}$.

We choose sections Y_0 and Y_1 of $\mathcal{O}(H)$ and $\mathcal{O}(H_\infty)$, respectively, such that they form a system of homogeneous coordinates on each fiber of $\varpi : \hat{W} \to \Sigma_2$. Then the equation of S' can be written as

$$\psi_0\zeta^2 Y_0^3 + \psi_1\zeta Y_0^2 Y_1 + \psi_2\zeta Y_0 Y_1^2 + \psi_3 Y_1^3 = 0, \tag{8}$$

where ζ, ψ_0, ψ_1, ψ_2 and ψ_3 are sections on Σ_2 of $\mathcal{O}(\Delta_0)$, $\mathcal{O}(\Gamma)$, $\mathcal{O}(2\Delta_0{+}4\Gamma)$, $\mathcal{O}(3\Delta_0{+}7\Gamma)$ and $\mathcal{O}(5\Delta_0 + 10\Gamma)$, respectively. Thus S' is singular along the rational curve Λ defined by $\zeta = Y_1 = 0$. Since $(5\Delta_0 + 10\Gamma)\Delta_0 = 0$ on Σ_2, ψ_3 is constant on Δ_0. Since S' is irreducible, ψ_3 does not vanish on Δ_0. We also remark that ψ_0 is not identically zero. Thus the multiplicity of S' at a generic point of Λ is 2. Moreover, S' has only isolated singular points except for the double curve Λ.

Let $\tau : \tilde{W} \to \hat{W}$ be the blowing-up with center Λ and put $\mathcal{E} = \tau^{-1}(\Lambda)$. If $\tilde{S}$ is the proper transform of S' by τ, then it is linearly equivalent to $\tau^*S' - 2\mathcal{E}$. Since the normal sheaf of $\Lambda \simeq \mathbf{P}^1$ in $\hat{W}$ is $\mathcal{O}(-1) \oplus \mathcal{O}(-2)$, $\mathcal{E}$ is isomorphic to Σ_1.

7.2 We show that $\tilde{S}$ is nonsingular in a neighbourhood of $\tilde{S} \cap \mathcal{E}$. For any point $P \in \Lambda$, we can assume that (ζ, y, t) forms a local coordinate system in a neighbourhood U of P, where $y = Y_1/Y_0$ and t is a local parameter of Λ at P. We regard the ψ_i in (8) as functions of (t, ζ) in U. Then, on U, S' is defined by

$$\psi_3 y^3 + \psi_2\zeta y^2 + \psi_1\zeta y + \psi_0\zeta^2 = 0.$$

Since ψ_3 does not vanish in a neighbourhood of Λ, we can assume $\psi_2 = 0$ by replacing y by $y + \psi_2\zeta/(3\psi_3)$.

We cover $\tau^{-1}(U)$ by open sets U_1, U_2 with coordinates (u_1, v_1, t_1), (u_2, v_2, t_2), respectively, where

$$\zeta = u_1 v_1 = u_2, \ \ y = u_1 = u_2 v_2, \ \ t = t_1 = t_2.$$

Then $\tilde{S}$ is defined by

$$\begin{aligned} \psi_3 u_1 + \psi_1 v_1 + \psi_0 v_1^2 &= 0 \quad \text{on } U_1, \\ \psi_3 u_2 v_2^3 + \psi_1 v_2 + \psi_0 &= 0 \quad \text{on } U_2. \end{aligned} \tag{9}$$

Since ψ_3 is a nonzero constant on $v_1 = 0$, $\tilde{S}$ is nonsingular in U_1. Since ψ_0 is at most of multiplicity 1 at P, $\tilde{S}$ is nonsingular in U_2.

Thus $\tilde{S}$ has only isolated singularities. Hence it is normal by Serre's criterion. We remark that $\tilde{S}$ is not necessarily the normalization of S': If ψ_1 does not vanish identically on Δ_0, then $\mathcal{E} \cap \tilde{S}$ consists of two disjoint curves E and $\tilde{G}$, where E is defined by $v_1 = 0$, and $\tilde{G}$ is defined by $\psi_1 + \psi_0 v_1 = 0$ on U_1 and $\psi_1 v_2 + \psi_0 = 0$ on

U_2. Thus $\tilde{S}$ is the normalization of S' in this case. On the other hand, if ψ_1 vanishes identically on Δ_0, then we have $\mathcal{E}|_{\tilde{S}} = 2E + \hat{G}$, where $\hat{G}$ is defined by $\psi_0 = 0$. Since ψ_0 has a simple zero on Δ_0, $\hat{G}$ is a fiber of $\mathcal{E} \to \Lambda$. Thus $\tilde{S}$ is obtained by blowing up a point on the normalization of S'.

We show that E is a (-1)-curve on $\tilde{S}$. We let Z be the divisor on $\hat{W}$ defined by $\varpi^*\zeta$ and $\tilde{Z}$ its proper transform by τ. Then $\tilde{Z} \sim \tau^*\varpi^*\Delta_0 - \mathcal{E}$ and it is given by $v_1 = 0$. Thus (9) shows that $E = \tilde{Z}|_{\tilde{S}}$. We identify $\tilde{Z}$ with Σ_1. Then we have

$$\tau^*H|_{\tilde{Z}} \sim \Delta_0 + \Gamma,\ \tau^*\varpi^*\Gamma|_{\tilde{Z}} \sim \Gamma,\ \mathcal{E}|_{\tilde{Z}} \sim \Delta_0.$$

Further we have

$$\begin{aligned} \tilde{Z}|_{\tilde{Z}} &\sim (\tau^*\varpi^*(\Delta_0 + 2\Gamma) - 2\tau^*\varpi^*\Gamma - \mathcal{E})|_{\tilde{Z}} \\ &\sim -\Delta_0 - 2\Gamma, \end{aligned}$$

since we have $\Delta_0(\Delta_0 + 2\Gamma) = 0$ on Σ_2. Thus we get

$$\begin{aligned} E^2 &= \tilde{Z}^2\tilde{S} \\ &= \tilde{Z}^2[\tau^*(3H + \varpi^*\Gamma) + 2\tilde{Z}] \\ &= -(\Delta_0 + 2\Gamma)[3(\Delta_0 + \Gamma) + \Gamma - 2(\Delta_0 + 2\Gamma)] \quad (\text{on } \tilde{Z} \simeq \Sigma_1) \\ &= -1. \end{aligned}$$

Lemma 7.3 *The map $\hat{f} : \hat{S} \to S'$ factors through $\tilde{S}$.*

Proof. Let $\mathcal{I}$ be the ideal sheaf of Λ in S'. We show that $\hat{f}^{-1}\mathcal{I} \cdot \mathcal{O}_{\hat{S}}$ is invertible. For this purpose, we freely use the notation of 5.2. Let $\mathcal{G}$ be the greatest common divisor of $\hat{f}^*H_\infty$ and $\hat{f}^*\varpi^*\Delta_0 = \hat{\mu}^*\Delta_0$. Recall that $\hat{f}^*H_\infty = \sigma^*G + E$ and that σ^*G can be written as $G_0 + G_1 + G'$ with $\hat{\mu}(G_0) = \Delta_0$, $\hat{\mu}(G_1) \sim \Gamma$. Since $\hat{\mu}(E) = \Delta_0$, we have $\mathcal{G} \geq E$ and $\mathcal{G} \geq G_0$. Thus it suffices for us to show that $\hat{f}^*H_\infty - \mathcal{G}$ and $\hat{\mu}^*\Delta_0 - \mathcal{G}$ do not meet.

Let G_2 be an irreducible component of G'. Since $\hat{\mu}(G_2)$ is a point, we have $G_2(\hat{\mu}^*\Delta_0) = 0$. Thus, if G_2 intersects with a component of $\hat{\mu}^*\Delta_0$, it must be a component of $\mathcal{G}$. We next consider the curve G_1. Clearly, it is not a component of $\hat{\mu}^*\Delta_0$. Since $G_1(\hat{\mu}^*\Delta_0) = 1$, it suffices to show $G_1\mathcal{G} > 0$.

The case $G_0 \neq E$: We have $\mathcal{G} \geq E + G_0$. Since E is a (-1)-curve, we have

$$-1 = \hat{K}E = ME + 2E^2 + E(\sigma^*G). \tag{10}$$

Thus $E(\sigma^*G) = 0$. Then each irreducible component of σ^*G is a (-2)-curve. In particular, $\hat{K}G_1 = 0$ and we have $G_0G_1 + G'G_1 = 1$. If $G_0G_1 = 1$, then we get $G_1\mathcal{G} > 0$ since $\mathcal{G} \geq E + G_0$. We assume $G_0G_1 = 0$ and $G'G_1 = 1$. Let G_2 be the component of G' with $G_1G_2 = 1$. If it is a component of $\mathcal{G}$, then we are done. So we assume that G_2

is not a component of $\mathcal{G}$ and show that this leads us to a contradiction. Since G_2 is a (-2)-curve, we have

$$0 = \hat{K}G_2 = MG_2 + 2EG_2 + G_0G_2 + G_1G_2 + G_2^2 + G_2(G' - G_2).$$

Since $MG_2 = EG_2 = G_0G_2 = 0$, we have $G_2(G' - G_2) = 1$. Thus we can find a component G_3 of G' such that $G_2G_3 = 1$. Since G_2 is not a component of $\mathcal{G}$, so is G_3. Then we have, as above, $G_3(G' - G_2 - G_3) = 1$ since G_3 is a (-2)-curve. Thus we can find a component G_4 of $G' - G_2 - G_3$ with $G_3G_4 = 1$, which is not a component of $\mathcal{G}$. Continuing this procedure, we would get infinite number of components G_i, $i \geq 2$, of G'. This is impossible.

The case $G_0 = E$: We have $E(G_1 + G') = 1$ by (10). If $EG_1 = 1$, then we are done. So we assume that $EG_1 = 0$ and $EG' = 1$. Then G_1 is a (-2)-curve. Thus, as in the previous case, we can find a component G_2 of G' with $G_1G_2 = 1$. If G_2 is a component of $\mathcal{G}$, we are done. So we assume that this is not the case. We in particular have $G_2E = 0$. Thus it is a (-2)-curve and we can find a component G_3 of $G' - G_2$ with $G_2G_3 = 1$. Then, as in the previous case, we get a contradiction by continuing this procedure. Thus we have a chain $\mathcal{G}' = G_1 + G_2 + \cdots + G_n \leq \mathcal{G}$ of irreducible components of σ^*G such that $G_iG_{i+1} = 1$, G_i is a (-2)-curve for $i < n$, and G_n is a (-3)-curve. *q.e.d.*

7.4 We calculate the invariants of $\tilde{S}$. Since $\tau^*(K_{\hat{W}} + S' - \mathcal{E}) \sim \tau^*H + \tilde{Z}$, the dualizing sheaf of $\tilde{S}$ is given by $\omega_{\tilde{S}} = \mathcal{O}(\tau^*H|_{\tilde{S}} + E)$. Since $E(\tau^*H) = 0$ and $E^2 = -1$, we get $\omega_{\tilde{S}}^2 = 14$ by

$$\begin{aligned}
(\tau^*H|_{\tilde{S}})^2 &= (\tau^*H)^2(\tau^*(3H + \varpi^*(2\Delta_0 + \Gamma)) - 2\mathcal{E}) \\
&= H^2(3H + \varpi^*(2\Delta_0 + \Gamma)) \qquad (\text{on } \hat{W}) \\
&= 15,
\end{aligned}$$

where we have used the equality $H^2 = H\varpi^*(\Delta_0 + 3\Gamma)$ in the Chow ring of $\hat{W}$.

In order to calculate $\chi(\mathcal{O}_{\tilde{S}})$, we use the exact sequences

$$0 \to \mathcal{O}(K_{\tilde{W}}) \to \mathcal{O}(\tau^*H + \tilde{Z}) \to \omega_{\tilde{S}} \to 0,$$

and

$$0 \to \mathcal{O}(\tau^*H) \to \mathcal{O}(\tau^*H + \tilde{Z}) \to \mathcal{O}_{\tilde{Z}}(-\Gamma) \to 0.$$

From these, we get $H^q(\omega_{\tilde{S}}) \simeq H^q(\mathcal{O}(\tau^*H + \tilde{Z})) \simeq H^q(\mathcal{O}(\tau^*H)) \simeq H^q(\hat{W}, \mathcal{O}(H))$ for $q < 2$. Thus $h^0(\omega_{\tilde{S}}) = 7$, $h^1(\omega_{\tilde{S}}) = 0$. Since $\tilde{S}$ is a normal surface on the nonsingular threefold $\tilde{W}$, we see that $\tilde{S}$ has only RDP as its singularity from the equality $\chi(\mathcal{O}_{\hat{S}}) = \chi(\mathcal{O}_{\tilde{S}})$.

7.5 Conversely, we start from S' defined by (8). We can assume that it has only RDP except for the double curve Λ if we choose the ψ_i in (8) general. Let $\tilde{S}$ be its proper transform by $\tau : \tilde{W} \to \hat{W}$ described above. Then it has only RDP. Let $\hat{S}$ be the minimal resolution of $\tilde{S}$. Then $\hat{S}$ has a (-1)-curve E. Let $\sigma : \hat{S} \to S$ be the contraction of E. Then S is a minimal surface satisfying $p_g = 7$, $q = 0$ and $c_1^2 = 15$. Since $H^0(K)$ is in bijection with $H^0(\hat{W}, \mathcal{O}(H))$, it is of type I.

8 Surfaces of type I-0: The case (s.2).

In this section, we study the case (s.2) of 6.1.

8.1 Since $p_g = 6$, we have $(a,\ b,\ c) = (0,\ 1,\ 2)$ by Lemma 6.3. As in 7.1, we can show that $S' = \hat{f}(\hat{S})$ is linearly equivalent to $3H + \varpi^*(2\Delta_0 + \Gamma)$ on $\hat{W} = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(\Delta_0 + 2\Gamma))$ and that its equation can be written as

$$\psi_0 \zeta Y_0^3 + \psi_1 Y_0^2 Y_1 + \psi_2 Y_0 Y_1^2 + \psi_3 Y_1^3 = 0, \tag{11}$$

where $(Y_0,\ Y_1)$ is a system of homogeneous coordinates on fibers of $\varpi : \hat{W} \to \Sigma_1$, and ζ, ψ_0 and ψ_i, $1 \leq i \leq 3$, are sections of $\mathcal{O}(\Delta_0)$, $\mathcal{O}(\Delta_0 + \Gamma)$ and $\mathcal{O}((i+2)\Delta_0 + (2i+1)\Gamma)$, respectively. We let Z be the divisor on $\hat{W}$ defined by ζ.

We study the singular locus of S'. For this purpose, we let $H \in |H|$ be a general member. Then it is isomorphic to Σ_1 and $H \cap S'$ can be identified with C^*. Since $p_a(C^*) = g(C) + 1$, C^* has a singular point P of multiplicity 2. By Lemma 6.6, $P \in H \cap Z$. If we vary H in $|H|$, such P traces a curve Λ on $Z \simeq \Sigma_1$, which is a multiple curve of S'. Putting $\zeta = 0$ in (11), we get $Y_1(\psi_1 Y_0^2 + \psi_2 Y_0 Y_1 + \psi_3 Y_1^2) = 0$. Thus, by identifying Z with Σ_1, we get $S'|_Z = \Delta_0 + G'$ with $G' \sim 2\Delta_0 + 2\Gamma$. Since $G' \geq 2\Lambda$, we have either

i) $S'|_Z = \Delta_0 + 2\Lambda$, $\Lambda \sim \Delta_0 + \Gamma$ is irreducible, or

ii) $S'|_Z = 3\Delta_0 + 2\Lambda$, $\Lambda \sim \Gamma$.

We let $\hat{D}$ be a general member of $|\hat{\mu}^*\Gamma|$ on $\hat{S}$, and put $D' = \hat{f}(\hat{D})$. Then $D' \sim \varpi^*\Gamma|_{S'}$. Since $\omega_{D'} = \mathcal{O}_{D'}(H + \varpi^*(\Delta_0 + \Gamma))$, we have

$$\deg \omega_{D'} = (H + \varpi^*(\Delta_0 + \Gamma))(3H + \varpi^*(2\Delta_0 + \Gamma))\varpi^*\Gamma = 8.$$

Thus D' is of arithmetic genus 5. Since $\hat{D}$ is of genus 4, D' has a singular point of multiplicity 2. Thus S' has a multiple curve which intersects with D'. In the case i), Λ has such a property. In the case ii), there is a multiple curve other than Λ.

8.2 Here we consider the case i) of 8.1. Since a general $H \in |H|$ induces on $Z \simeq \Sigma_1$ an irreducible divisor linearly equivalent to $\Delta_0 + \Gamma$ and since the restriction map $H^0(\hat{W}, \mathcal{O}(H)) \to H^0(Z, \mathcal{O}(H|_Z))$ is surjective, we can assume that $\Lambda = Z \cap (Y_0)$ by a suitable change of $(Y_0) \in |H|$. Thus ψ_2 and ψ_3 in (11) can be divided by ζ. Since Λ is a double curve, S' is defined by

$$\psi_0 \zeta Y_0^3 + \psi_1 Y_0^2 Y_1 + \psi_2' \zeta Y_0 Y_1^2 + \psi_3' \zeta^2 Y_1^3 = 0, \tag{12}$$

where ψ_i', $i = 2, 3$, is a section of $\mathcal{O}(3\Delta_0 + (2i+1)\Gamma)$ on Σ_1. We remark that ψ_1 must be a nonzero constant on Δ_0, since S' is irreducible. In particular, the multiplicity of S' along Λ is 2. Further, we can assume that ψ_2' and ψ_3' do not vanish identically on Δ_0. This can be seen as follows: Let ω be a section of $\mathcal{O}(2\Gamma)$ and put $\phi = \zeta\omega$. If we replace Y_0 by $Y_0 + \phi Y_1$, then ψ_3', for example, is replaced by $\psi_0 \zeta^2 \omega^3 + \psi_1 \omega^2 + \psi_2' \zeta \omega + \psi_3'$. Thus, by such a change of coordinates, we can assume that ψ_3' does not vanish identically on Δ_0, since $\psi_1 \neq 0$ on Δ_0.

Let $\tau : \tilde{W} \to \hat{W}$ be the blowing-up with center Λ and put $\mathcal{E} = \tau^{-1}(\Lambda)$. Let $\tilde{S}$ be the proper transform of S' by τ. Then $\tilde{S} \sim \tau^* S' - 2\mathcal{E}$. For any point $Q \in \Lambda$, we can assume that $(\zeta,\ y,\ t)$ forms a local coordinate system in a neighbourhood U of Q, where $y = Y_0/Y_1$ and t is a local parameter of Λ at Q. We cover $\tau^{-1}(U)$ by open sets U_1, U_2 with coordinates $(u_1,\ v_1,\ t_1)$, $(u_2,\ v_2,\ t_2)$, respectively, where

$$\zeta = u_1 v_1 = u_2,\ \ y = u_1 = u_2 v_2,\ \ t = t_1 = t_2.$$

Then $\tilde{S}$ is defined by

$$\begin{aligned} \psi_0 u_1^2 v_1 + \psi_1 + \psi_2' v_1 + \psi_3' v_1^2 &= 0 \quad \text{on } U_1, \\ \psi_0 u_2^2 v_2^3 + \psi_1 v_2^2 + \psi_2' v_2 + \psi_3' &= 0 \quad \text{on } U_2. \end{aligned}$$

Since ψ_1 is a nonzero constant on Δ_0, $\tilde{S}$ does not meet the divisor (v_1) on U_1. Since ψ_2' and ψ_3' do not vanish identically on Δ_0, $\tilde{S}$ has at most isolated singularities on U_2. The curve $\mathcal{E} \cap \tilde{S}$ is defined by $u_1 = \psi_1 + \psi_2' v_1 + \psi_3' v_1^2 = 0$ on U_1 and $u_2 = \psi_1 v_2^2 + \psi_2' v_2 + \psi_3' = 0$ on U_2. Thus $\tilde{S}$ is the normalization of S'. In particular, the map $\hat{f} : \hat{S} \to S'$ factors through $\tilde{S}$.

Let $\tilde{Z}$ be the proper transform of Z. Then we have $\omega_{\tilde{S}} = \mathcal{O}_{\tilde{S}}(\tau^* H + \tilde{Z})$. We recall that S' contains a rational curve E defined by $\zeta = Y_1 = 0$ and that it is nonsingular in a neighbourhood of E. Thus the above local calculation shows in particular that $\tilde{Z}$ induces E on $\tilde{S}$. We have the following by identifying $\tilde{Z}$ with Σ_1.

$$\tau^* H|_{\tilde{Z}} \sim \mathcal{E}|_{\tilde{Z}} \sim \Delta_0 + \Gamma,\ \ \tau^* \varpi^* \Gamma|_{\tilde{Z}} \sim \Gamma,\ \ \tilde{Z}|_{\tilde{Z}} \sim -\Delta_0 - 2\Gamma.$$

Then

$$E^2 = \tilde{Z}^2 \tilde{S} = \tilde{Z}^2 [\tau^*(3H + \varpi^* \Gamma) + 2\tilde{Z}] = -1.$$

Thus E is a (-1)-curve. The invariants of $\tilde{S}$ can be calculated as in 7.4. We get $\omega_{\tilde{S}}^2 = 11$, $h^0(\omega_{\tilde{S}}) = 6$ and $h^1(\omega_{\tilde{S}}) = 0$. We remark that $H^0(\omega_{\tilde{S}})$ is in bijection with $H^0(\hat{W}, \mathcal{O}(H))$. Since $\chi(\mathcal{O}_{\hat{S}}) = \chi(\mathcal{O}_{\tilde{S}})$, we see that $\tilde{S}$ has only RDP. The existence of surfaces of this type is now clear.

8.3 We consider the case ii) of 8.1. In this case, ψ_i, $1 \le i \le 2$, can be divided by ζ. We let P be the point on Σ_1 with $\varpi^{-1}(P) = \Lambda$. Then P lies on the zero section Δ_0. Since Λ is a double curve, the equation of S' can be rewritten as

$$\phi_0\zeta^2 Y_0^3 + \phi_1\zeta Y_0^2 Y_1 + \phi_2\zeta Y_0 Y_1^2 + \phi_3 Y_1^3 = 0, \tag{13}$$

where ϕ_0, ϕ_1, ϕ_2 and ϕ_3 are sections of $\mathcal{O}(\Gamma)$, $\mathcal{O}(2\Delta_0+3\Gamma)$, $\mathcal{O}(3\Delta_0+5\Gamma)$ and $\mathcal{O}(5\Delta_0+7\Gamma)$ on Σ_1, respectively, which satisfy the conditions:

0) ϕ_0 is not identically zero.

1) ϕ_1 and ϕ_2 vanish at P.

2) P is a singular point of multiplicity 2 of the curve (ϕ_3).

Thus, if ζ_0 is a section of $\mathcal{O}(\Delta_0+\Gamma)$ such that (ζ_0, ζ) forms a system of homogeneous coordinates on fibers of Σ_1, then ϕ_i can be written as

$$\begin{aligned}
\phi_1 &= \alpha_0\omega\zeta_0^2 + \alpha_1\zeta_0\zeta + \alpha_2\zeta^2, \\
\phi_2 &= \beta_0\omega\zeta_0^3 + \beta_1\zeta_0^2\zeta + \beta_2\zeta_0\zeta^2 + \beta_3\zeta^3, \\
\phi_3 &= \omega^2\zeta_0^5 + \gamma_1\omega\zeta_0^4\zeta + \gamma_2\zeta_0^3\zeta^2 + \gamma_3\zeta_0^2\zeta^3 + \gamma_4\zeta_0\zeta^4 + \gamma_5\zeta^5,
\end{aligned}$$

where ω is a linear form on $\mathbf{P}^1$ which gives P on Δ_0, and α_i, β_j, γ_k are homogeneous forms on $\mathbf{P}^1$ whose degree can be determined uniquely by the linear equivalence classes of ϕ's. In particular, α_0 is a constant. Further, we can assume that (ϕ_3) is irreducible and nonsingular except for P.

We let Λ_0 be the curve on $\hat{W}$ defined by $\zeta = Y_1 = 0$. Then S' has only isolated singular points except for the curve $\Lambda + \Lambda_0$. We let $\tau_1 : W_1 \to \hat{W}$ be the blowing-up with center Λ_0 and denote by $\mathcal{E}'$ the exceptional divisor. Since the normal sheaf of $\Lambda_0 \simeq \mathbf{P}^1$ in $\hat{W}$ is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, $\mathcal{E}'$ is isomorphic to Σ_0. If we denote by S'' the proper transform of S' by τ_1, then we can show that $\hat{f} : \hat{S} \to S'$ factors through it as in Lemma 7.3.

For any point $Q \in \Lambda_0$, we can assume that (ζ, y, t) forms a local coordinate system in a neighbourhood U of Q, where $y = Y_1/Y_0$ and t is a local parameter of Λ_0 at Q. We cover $\tau^{-1}(U)$ by open sets U_1, U_2 with coordinates (u_1, v_1, t_1), (u_2, v_2, t_2), respectively, where

$$\zeta = u_1 v_1 = u_2, \quad y = u_1 = u_2 v_2, \quad t = t_1 = t_2.$$

Then S'' is defined by

$$\begin{aligned}
\phi_0 v_1^2 + \phi_1 v_1 + \phi_2 u_1 v_1 + \phi_3 u_1 &= 0 \quad \text{on } U_1, \\
\phi_0 + \phi_1 v_2 + \phi_2 u_2 v_2^2 + \phi_3 u_2 v_2^3 &= 0 \quad \text{on } U_2.
\end{aligned}$$

Since ϕ_0 is the function in t of degree 1, S'' is nonsingular on U_2. On U_1, it is singular along the curve $v_1 = \omega = 0$, which is the proper transform Λ' of Λ by τ_1. $S'' \cap \mathcal{E}'$ is defined by $u_1 = v_1(\phi_0 v_1 + \phi_1) = 0$ on U_1 and $u_2 = \phi_0 + \phi_1 v_2 = 0$ on U_2.

Let $\tau_2 : \tilde{W} \to W_1$ be the blowing-up with center Λ'. Let $\mathcal{E}_1$ be the proper transform of $\mathcal{E}'$ by τ_2, and let $\mathcal{E}_2$ be the exceptional divisor of τ_2. Then $\mathcal{E}_1$ is Σ_0 blown up at one point, and $\mathcal{E}_2$ is isomorphic to Σ_1. The proper transform $\tilde{S}$ of S' by $\tau = \tau_1 \circ \tau_2$ is linearly equivalent to $\tau^* S' - 2(\mathcal{E}_1 + \mathcal{E}_2)$.

We let V be a sufficiently small open neighbourhood in W_1 of a point $Q \in \Lambda'$. On V, we can regard ϕ_0, α_i, β_j, γ_k as functions of ω. In particular, we can write $\phi_0 = r_0\omega + r_1$ with some constants r_0, r_1. If $Q \in \mathcal{E}'$, then we can assume that (u_1, v_1, ω) is a local coordinate system on V. We cover $\tau_2^{-1}(V)$ by open sets V_1, V_2 with coordinates (u_1, w_1, x_1) and (u_1, w_2, x_2), respectively, where

$$\omega = w_1 x_1 = w_2, \quad v_1 = w_1 = w_2 x_2.$$

Then $\tilde{S}$ is defined by

$$\begin{aligned} &\phi_0 + \phi_1' + \phi_2' u_1 + \phi_3' u_1 = 0 && \text{on } V_1,\\ &\phi_0 x_2^2 + \phi_1'' x_2 + \phi_2'' u_1 x_2 + \phi_3'' u_1 = 0 && \text{on } V_2. \end{aligned}$$

where

$$\begin{aligned} \phi_1' &= \alpha_0 x_1 + \alpha_1 u_1 + \alpha_2 u_1^2 w_1,\\ \phi_2' &= \beta_0 x_1 + \beta_1 u_1 + \beta_2 u_1^2 w_1 + \beta_3 u_1^3 w_1^2,\\ \phi_3' &= x_1^2 + \gamma_1 u_1 x_1 + \textstyle\sum_{i=2}^5 \gamma_i u_1^i w_1^{i-2},\\ \phi_1'' &= \alpha_0 + \alpha_1 u_1 x_2 + \alpha_2 u_1^2 w_2 x_2^2,\\ \phi_2'' &= \beta_0 + \beta_1 u_1 x_2 + \beta_2 (u_1 x_2)^2 w_2 + \beta_3 (u_1 x_2)^3 w_2^2,\\ \phi_3'' &= 1 + \gamma_1 u_1 x_2 + \textstyle\sum_{i=2}^5 \gamma_i (u_1 x_2)^i w_2^{i-2}. \end{aligned}$$

Thus $\tilde{S}$ is nonsingular on V_2 and it has at most isolated singular points on V_1. Varying Q on Λ', we find similarly that $\tilde{S}$ has only isolated singular points. $\tilde{S} \cap \mathcal{E}_1$ is defined by $u_2 = r_0\omega + r_1 + \alpha_0 \omega v_2 = 0$ on U_2, $u_1 = r_0 w_1 x_1 + r_1 + \alpha_0 x_1 = 0$ on V_1 and $u_1 = x_2((r_0 w_2 + r_1)x_2 + \alpha_0) = 0$ on V_2. $\tilde{S} \cap \mathcal{E}_2$ is defined by

$$\begin{aligned} &w_1 = r_1 + \alpha_0 x_1 + u_1(\alpha_1 + \beta_0 x_1 + \beta_1 u_1 + x_1^2 + \gamma_1 u_1 x_1 + \gamma_2 u_1^2) = 0 && \text{on } V_1,\\ &w_2 = r_1 x_2^2 + \alpha_0 x_2 + u_1(\alpha_1 x_2^2 + \beta_0 x_2 + \beta_1 u_1 x_2^2 + 1 + \gamma_1 u_1 x_2 + \gamma_2 u_1^2 x_2^2) = 0 && \text{on } V_2. \end{aligned}$$

Thus $\tilde{S}$ is the normalization of S'' unless $r_1 = \alpha_0 = 0$. If $r_1 = \alpha_0 = 0$, then it is obtained by blowing up a point of the normalization.

Lemma 8.4 *The map $\hat{f} : \hat{S} \to S'$ factors through $\tilde{S}$.*

Proof. We can assume $r_1 = \alpha_0 = 0$. Let Γ_ω be the fiber of Σ_1 defined by $\omega = 0$. Let us use the notation of 5.2 and the proof of Lemma 7.3. We have the natural map $\hat{\mu} : \hat{S} \to \Sigma_1$. We let $\mathcal{F}$ be the greatest common divisor of $\hat{\mu}^*\Gamma_\omega$, $\hat{\mu}^*\Delta_0$ and $\hat{f}^* H_\infty$. As we have seen above, $\tilde{S}$ is obtained as the blowing up of a point of the normalization of S''. Thus it suffices to show that $\mathcal{F}$ is not zero and that $\hat{\mu}^*\Gamma_\omega - \mathcal{F}$, $\hat{\mu}^*\Delta_0 - \mathcal{F}$ and $\hat{f}^* H_\infty - \mathcal{F}$ do not meet simultaneously.

We first extract information from $\hat{S} \to S''$. Since $U_2 \cap S'' \cap (v_2)$ is defined by $\phi_0 = 0$, G_1 is a common component of $\hat{f}^* H_\infty$ and $\hat{\mu}^* \Gamma_\omega$. Similarly, considering $S'' \cap \mathcal{E}'$, we can find a common component G_2 of them, which results from the curve $\mathcal{E}' \cap \tau_1^* \varpi^* \Gamma_\omega$ and satisfies $G_1 G_2 = 1$. G_1 meets no other component of $\hat{f}^* H_\infty$. In particular, G_1 does not meet E. Thus it is a (-2)-curve. Further, it can be checked that $G_0 = E$. By the proof of Lemma 7.3, we have a chain of rational curves $\mathcal{C} = G_2 + \cdots + G_n$ consisting of components of $\sigma^* G$ satisfying $G_i G_{i+1} = 1$, $i < n$, $G_i G_j = 0$ for $|i - j| > 1$, G_i is a (-2)-curve for $i < n$, and G_n is a (-3)-curve with $G_n E = 1$. Note that G_i is the unique irreducible component of $\sigma^* G$ with $G_i G_{i-1} > 0$ and that the multiplicity of G_i in $\sigma^* G$ is one. We have $\mathcal{C} \leq \mathcal{G}$. Since the support of $\tau_1^* H_\infty \cap S''$ is contained in the union of $\tau_1^* \varpi^* \Gamma_\omega$ and the proper transform Z' of Z by τ_1, $\mathcal{C} - G_2$ results from the point $\tau_1^* H_\infty \cap Z' \cap \tau_1^* \varpi^* \Gamma_\omega$. Thus it follows that $G_1 + \mathcal{C}$ coinsides with the greatest common divisor of $\hat{f}^* H_\infty$ and $\hat{\mu}^* \Gamma_\omega$. Therefore $\mathcal{F} = \mathcal{C}$.

Thus it suffices for our purpose to show that any irreducible component G^* of $G'' := \sigma^* G - E - G_1 - \mathcal{C}$ does not meet $\mathcal{C}$. The curve G_i, $i < n$, does not meet G^*. We show G_n also does not meet it. Since G_n is a (-3)-curve, we have $1 = \hat{K} G_n = (M + 3E + G_1 + \mathcal{C} + G'') G_n = 3 + 1 - 3 + G'' G_n$. Thus $G'' G_n = 0$. This show the assertion. *q.e.d.*

8.5 We let $\tilde{Z}$ be the proper transform of Z by τ. Then $\tilde{Z} \sim \tau^* \varpi^* \Delta_0 - \mathcal{E}_1 - \mathcal{E}_2$. Thus $\omega_{\tilde{S}} = \mathcal{O}_{\tilde{S}}(\tau^* H + \tilde{Z})$. We show that $E := \tilde{Z}|_{\tilde{S}}$ is a (-1)-curve. For this purpose, we identify $\tilde{Z}$ with Σ_1. Then we have

$$\tau^* H|_{\tilde{Z}} \sim \Delta_0 + \Gamma,\ \mathcal{E}_1|_{\tilde{Z}} \sim \Delta_0,\ \mathcal{E}_2|_{\tilde{Z}} \sim \Gamma.$$

Since $\tilde{Z}|_{\tilde{Z}} \sim (\tau^* \varpi^* (\Delta_0 + \Gamma) - \tau^* \varpi^* \Gamma - \mathcal{E}_1 - \mathcal{E}_2)|_{\tilde{Z}} \sim -\Delta_0 - 2\Gamma$, we have $\tilde{S}|_{\tilde{Z}} \sim 3(\Delta_0 + \Gamma) + \Gamma + 2\tilde{Z}|_{\tilde{Z}} \sim \Delta_0$. Thus E is a nonsingular rational curve. As we saw above, $\tilde{S}$ is nonsingular in a neighbourhood of E. Since

$$E^2 = \tilde{Z}^2 \tilde{S} = \tilde{Z}^2 (\tau^* (3H + \varpi^* \Gamma) + 2\tilde{Z}) = -1,$$

we conclude that E is a (-1)-curve. Thus $\omega_{\tilde{S}} \sim \tau^* H|_{\tilde{S}} + E$. The invariants of $\tilde{S}$ can be calculated as in 7.4. We have $\omega_{\tilde{S}}^2 = 11$, $h^0(\omega_{\tilde{S}}) = 6$ and $h^1(\omega_{\tilde{S}}) = 0$. Further, $H^0(\omega_{\tilde{S}})$ is in bijection with $H^0(\hat{W}, \mathcal{O}(H))$. In particular, since we have $\chi(\mathcal{O}_{\hat{S}}) = \chi(\mathcal{O}_{\tilde{S}})$, $\tilde{S}$ has only RDP. The existence of surfaces of this type can be checked, if we choose α_i, β_j, γ_k general.

9 Surfaces of type I-0: The case (s.3).

In this section, we study the case (s.3) of 6.1.

Lemma 9.1 *Suppose that $S^* \sim 4T - (p_g - 6)F$ on $W = \mathbf{P}_{a,b,c}$. Then*
1) the dualizing sheaf ω_{S^*} of S^* is given by $\omega_{S^*} = \mathcal{O}_{S^*}(T+F)$ and

$$h^0(S^*, \omega_{S^*}) = p_g + 3, \quad h^1(S^*, \omega_{S^*}) = 0,$$
$$\omega_{S^*}^2 = 3h^0(S^*, \omega_{S^*}) - 7 = 3p_g + 2,$$

2) the singular locus $\mathrm{Sing}(S^*)$ of S^* has no horizontal component, i.e., the image via $g : S^* \to \mathbf{P}^1$ of any connected component of $\mathrm{Sing}(S^*)$ is one point.

Proof. 1): Since $K_W = -3T + (p_g - 5)F$, we get $\omega_{S^*} = \mathcal{O}_{S^*}(T+F)$ by the adjunction formula. By using the exact sequence

$$0 \to \mathcal{O}(K_W) \to \mathcal{O}(T+F) \to \omega_{S^*} \to 0$$

and the fact that

$$H^q(W, \mathcal{O}(K_W)) = 0 \quad \text{for } q < 3,$$
$$H^q(W, \mathcal{O}(T+F)) \simeq H^q(\mathbf{P}^1, \mathcal{O}(a+1) \oplus \mathcal{O}(b+1) \oplus \mathcal{O}(c+1))$$

for $\forall q$, we get the desired formulae for $h^q(S^*, \omega_{S^*})$. Further, an easy calculation shows $\omega_{S^*}^2 = (T+F)^2(4T - (p_g - 6)F) = 3p_g + 2$. Thus we get 1).

2): Suppose that $\mathrm{Sing}(S^*)$ has an horizontal component. Then any fiber of g is a singular plane quartic curve. Thus the normalization of S^* has a pencil of curves of genus ≤ 2. This contradicts that it is birational to the type I surface S. Thus $\mathrm{Sing}(S^*)$ is vertical. *q.e.d.*

Lemma 9.2 *Let S^* be as in Lemma 9.1. Then the double curve of S^* is the unique fiber C_s of $g : S^* \to \mathbf{P}^1$. Its support is a (possibly singular) conic curve or a line viewed in the fiber $\mathbf{P}^2$ of W.*

Proof. It follows from Lemma 6.7 that $C^* \sim 4\Delta_0 + (p_g + 2d)\Gamma$ on Σ_d and that C^* has two singular points P_1 and P_2 of multiplicity 2 on the same fiber Γ_s. Thus C^* meets Γ_s at no other points. We regard Σ_d as a member of $|T|$, and let F_s be the fiber of $\pi : W \to \mathbf{P}^1$ which induces Γ_s on Σ_d. The rational curve Γ_s is a line in $F_s \simeq \mathbf{P}^2$. If we vary Σ_d in $|T|$, then, by 2) of Lemma 9.1, singular points of C^* traces a curve on F_s, which is a double curve of S^*. Since C^* is nonsingular except for P_1 and P_2, there is no other multiple curve. Thus the double curve is a line or a conic (in F_s) according to whether the singular points of C^* are infinitely near or not. *q.e.d.*

Lemma 9.3 *Let S^* be as in Lemma 9.1. Then the integers a, b, c satisfy the conditions*

1) $a + c \leq 3b + 3$, *and*

2) $b \leq 2a + 2$.

Proof. We take sections X_0, X_1 and X_2 of $\mathcal{O}(T - aF)$, $\mathcal{O}(T - bF)$ and $\mathcal{O}(T - cF)$, respectively, such that they form a system of homogeneous coordinates on each fiber of $W = \mathbf{P}_{a,b,c} \to \mathbf{P}^1$. Then the equation of S^* can be written as

$$\sum_{i,j \geq 0, i+j \leq 4} \phi_{ij} X_0^{4-i-j} X_1^i X_2^j = 0, \tag{14}$$

where ϕ_{ij} is a homogeneous form of degree $(4 - i - j)a + ib + jc - (p_g - 6)$ on $\mathbf{P}^1$. If $4b < p_g - 6$, then the left hand side of (14) can be divided by X_2, which is impossible since S^* is irreducible. Thus we get 1). If $3a + c < p_g - 6$, then the curve B defined by $X_1 = X_2 = 0$ is a multiple curve of S^*. This is impossible by Lemma 9.2. Thus we get $3a + c \geq p_g - 6$, i.e., $b \leq 2a + 3$. If $b = 2a + 3$, then ϕ_{10} and ϕ_{01} are constants. As we have seen above, ϕ_{01} cannot be zero. But then S^* is nonsingular along B. Since the double curve C_s meets B, this is impossible. Thus we get 2). *q.e.d.*

9.4 We call C_s in Lemma 9.2 the *singular fiber* of S^*. We have the following list of singular fibers (as plane quartics):

1) $C_s = 2L$, where L is a nonsingular conic.

2) $C_s = 2L$, $L = L_1 + L_2$, where L_1, L_2 are distinct lines.

3) $C_s = 4L$, where L is a line.

We let F_s be the fiber of $\pi : W = \mathbf{P}_{a,b,c} \to \mathbf{P}^1$ containing C_s as in the proof of Lemma 9.2. Take an affine coordinate t on $\mathbf{P}^1$ such that $t = 0$ defines F_s on W. Then the equation (14) can be rewritten as

$$\sum_{i=0}^{4c-p_g+6} \psi_i(X_0, X_1, X_2) t^i = 0, \tag{15}$$

where the ψ_i are homogeneous forms of degree 4 in X_0, X_1, X_2. Then ψ_0 defines C_s on $F_s \simeq \mathbf{P}^2$.

We denote by $\nu : \hat{W} \to W$ the blowing-up with center L, and let S' be the proper transform of S^* by ν. Let $\mathcal{I}$ be the ideal sheaf of L in S^*. Since $f^{-1}\mathcal{I} \cdot \mathcal{O}_S$ is clearly invertible, $f : S \to S^*$ factors through S'.

In the following, we shall sketch the curve D_s on S coming from C_s, which we call the *canonical degenerate fiber* of $h = g \circ f : S \to \mathbf{P}^1$.

9.5 We assume here that S^* has the singular fiber of type 1). For any $P \in L$, we can take a small open neighbourhood U of P in W with coordinates (t, x, y), where t is as in 9.4 and (x, y) is a local coordinate system on $F_s \simeq \mathbf{P}^2$ such that L is defined by $x = 0$. Then, on U, the equation (15) can be rewritten as

$$x^2 + f_1 xt + \sum_{i=2}^{4c-(p_g-6)} f_i t^i = 0,$$

where the $f_i = f_i(x, y)$ are holomorphic functions in x, y.

Let $\nu : \hat{W} \to W$ and S' be as in 9.4. We put $\mathcal{E} = \nu^{-1}(L)$. Since the normal sheaf N_L of $L \simeq \mathbf{P}^1$ is given by $\mathcal{O}(4) \oplus \mathcal{O}$, $\mathcal{E}$ is isomorphic to Σ_4. We show $\mathcal{E}|_{\mathcal{E}} \sim -\Delta_0$: Since $\det N_L = \mathcal{O}(4)$, we have $\mathcal{O}_L(K_W) = \mathcal{O}(-6)$. Then we get

$$K_{\mathcal{E}} \sim (\nu^* K_W + 2\mathcal{E})|_{\mathcal{E}} \sim -6\Gamma + 2\mathcal{E}|_{\mathcal{E}}.$$

On the other hand, we have $K_{\mathcal{E}} \sim -2\Delta_0 - 6\Gamma$. Thus we get $\mathcal{E}|_{\mathcal{E}} \sim -\Delta_0$.

We have $S' \sim \nu^* S^* - 2\mathcal{E} \sim \nu^*(4T - (p_g - 6)F) - 2\mathcal{E}$. Thus $S'|_{\mathcal{E}} \sim 2\Delta_0 + 8\Gamma$ on $\mathcal{E} \simeq \Sigma_4$. Let F' be the proper transform of F_s. Then $F' \sim \nu^* F - \mathcal{E}$, and it does not meet S'. Further, S' is the normalization of S^*. These can be seen as follows. We cover $\nu^{-1}(U)$ by open sets U_1, U_2 with coordinates (u_1, v_1, y_1) and (u_2, v_2, y_2), respectively, where

$$t = u_1 v_1 = u_2, \ x = u_1 = u_2 v_2, \ y = y_1 = y_2.$$

Then S' is defined by

$$\begin{aligned} 1 + f_1 v_1 + \textstyle\sum_{i \geq 2} f_i u_1^{i-2} v_1^i = 0 \quad &\text{on } U_1, \\ v_2^2 + f_1 v_2 + \textstyle\sum_{i \geq 2} f_i u_2^{i-2} = 0 \quad &\text{on } U_2. \end{aligned}$$

Since F' is defined by $v_1 = 0$, it does not meet S'. We remark that F' induces on $\mathcal{E}$ the 0-section Δ_0. Thus $\nu|_{S'} : S' \to S^*$ is finite. $\mathcal{E} \cap S'$ is defined by $u_1 = 1 + f_1(0, y_1)v_1 + f_2(0, y_1)v_1^2$ on U_1 and $u_2 = v_2^2 + f_1(0, y_2)v_2 + f_2(0, y_2) = 0$ on U_2. Thus there are the following possibilities:

i) $S'|_{\mathcal{E}}$ is irreducible.

ii) $S'|_{\mathcal{E}}$ is reduced and consists of two irreducible components each of which is linerly equivalent to $\Delta_0 + 4\Gamma$.

iii) $S'|_{\mathcal{E}} = 2L'$, where $L' \sim \Delta_0 + 4\Gamma$ is irreducible.

If S' has a multiple curve, then it must be contained in $\nu^* F_s$ as we saw in Lemma 9.2. Thus S' may not be normal when iii). Since the singularities of C^* are not infinitely near, and since $\nu^* T$ meets L', we see that L' cannot be a multiple curve. It follows that S' is normal. The cases i) and ii) occur when the singular points of C^* are ordinary double points. On the other hand, the case iii) occurs when they are simple cusps.

Since $(\nu^* T + F - \mathcal{E})|_{S'} \sim \nu^* T|_{S'}$, we have $\omega_{S'} = \mathcal{O}_{S'}(\nu^* T)$. Then

$$\omega_{S'}^2 = (\nu^* T)^2 (\nu^* S^* - 2\mathcal{E}) = T^2(4T - (p_g - 6)F) = 3p_g - 6.$$

In order to calculate $\chi(\mathcal{O}_{S'})$, we use the exact sequences

$$0 \to \mathcal{O}(K_{\hat{W}}) \to \mathcal{O}(\nu^*T + F') \to \omega_{S'} \to 0,$$
$$0 \to \mathcal{O}(\nu^*T) \to \mathcal{O}(\nu^*T + F') \to \mathcal{O}_{F'}(-1) \to 0,$$

where F' is identified with $\mathbf{P}^2$. From the cohomology exact sequence derived from the first sequence, we get $H^q(\omega_{S'}) \simeq H^q(\hat{W}, \mathcal{O}(\nu^*T + F'))$ for $q < 2$. From the second one, we get $H^p(\mathcal{O}(\nu^*T + F')) \simeq H^p(W, \mathcal{O}(T))$ for any p. Thus we have $h^0(\omega_{S'}) = p_g$, $h^1(\omega_{S'}) = 0$. Then, since $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{S'})$, we see that S' has only RDP. The canonical degenerate fiber D_s is in general a hyperelliptic curve of genus 3 as i) shows.

9.6 We consider the singular fiber of type 2). Let $P \in L$. If it is not $L_1 \cap L_2$, then we can study S^* in a neighbourhood of P as in 9.5. So we assume $P = L_1 \cap L_2$. We can take a small open neighbourhood U of P in W with coordinates $(t,\, x,\, y)$, where t is as in 9.4 and $(x,\, y)$ is an affine coordinate system on $F_s \simeq \mathbf{P}^2$ such that L_1 and L_2 are defined by $x = 0$ and $y = 0$, respectively. Then, on U, the equation (15) can be rewritten as

$$x^2y^2 + f_1xyt + \sum_{i=2}^{4c-(p_g-6)} f_it^i = 0,$$

where the $f_i = f_i(x, y)$ are polynomial functions in x, y.

Let $\tau_1 : W_1 \to W$ be the blowing-up with center L_1. We denote F' and $\mathcal{E}'$ the proper transform of F_s and the exceptional divisor of τ_1, respectively. We let S_1 be the proper transform of S^* by τ_1. We cover $\tau^{-1}(U)$ by open sets U_1, U_2 with coordinates (u_1, v_1, y_1) and (u_2, v_2, y_2), respectively, where

$$t = u_1v_1 = u_2,\ x = u_1 = u_2v_2,\ y = y_1 = y_2.$$

Then S_1 is defined by

$$y_1^2 + f_1y_1v_1 + \textstyle\sum_{i\geq 2} f_iu_1^{i-2}v_1^i = 0 \quad \text{on } U_1,$$
$$y_2^2v_2^2 + f_1y_2v_2 + \textstyle\sum_{i\geq 2} f_iu_2^{i-2} = 0 \quad \text{on } U_2.$$

Let $\tau_2 : W_2 \to W_1$ be the blowing-up with center the proper transform L_2' of L_2 by τ_1. Put $\tau = \tau_1 \circ \tau_2$ and let $\hat{F}$ and $\mathcal{E}_1$ be the proper transform by τ_2 of F' and $\mathcal{E}'$, respectively. We also put $\mathcal{E}_2 = \tau_2^{-1}(L_2')$. We remark that $\mathcal{E}_1$ is Σ_1 blown up at a point on Δ_0, and $\mathcal{E}_2$ is isomorphic to Σ_2. If S_2 is the proper transform of S^* by τ, then it is linearly equivalent to $\tau^*S^* - 2\mathcal{E}_1 - 2\mathcal{E}_2$. Let V be a open neighbourhood of $Q \in F' \cap \mathcal{E}'$. We can assume (u_1, v_1, y_1) forms a system of local coordinates on V. We cover $\tau_2^{-1}(V)$ by two open sets V_1, V_2 with coordinates (u_1, w_1, x_1) and (u_1, w_2, x_2), respectively, where

$$v_1 = w_1x_1 = w_2,\ y_1 = w_1 = w_2x_2.$$

Then S_2 is defined by

$$
\begin{array}{ll}
1 + f_1 x_1 + \sum_{i\geq 2} f_i (u_1 w_1)^{i-2} x_1^i = 0 & \text{on } V_1, \\
x_2^2 + f_1 x_2 + \sum_{i\geq 2} f_i u_1^{i-2} w_2^{i-2} = 0 & \text{on } V_2.
\end{array}
$$

Thus it does not meet $\hat{F}$. $S_2 \cap \mathcal{E}_1$ is defined by $u_2 = y_2^2 v_2^2 + f_1(0, y_2) y_2 v_2 + f_2(0, y_2) = 0$ on U_2, $u_1 = 1 + f_1(0, w_1) x_1 + f_2(0, w_1) x_1^2 = 0$ on V_1 and $u_1 = x_2^2 + f_1(0, w_2 x_2) x_2 + f_2(0, w_2 x_2) = 0$ on V_2. Further, $S_2 \cap \mathcal{E}_2$ is defined by $w_1 = 1 + f_1(u_1, 0) x_1 + f_2(u_1, 0) x_1^2 = 0$ on V_1 and $w_2 = x_2^2 + f_1(u_1, 0) x_2 + f_2(u_1, 0) = 0$ on V_2. Since $\hat{F} \sim \tau^* F - \mathcal{E}_1 - \mathcal{E}_2$, and since it does not intersect with S_2, we have $\omega_{S_2} = \mathcal{O}_{S_2}(\tau^* T)$. Then we can calculate the invariants of S_2 as in 9.5. We have $\omega_{S_2}^2 = 3p_g - 6$, $h^0(\omega_{S_2}) = p_g$ and $h^1(\omega_{S_2}) = 0$. Further, $H^0(\omega_{S_2})$ is in bijection with $H^0(W, \mathcal{O}(T))$. In particular, we have $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{S_2})$.

We assume that S_2 is not normal and show that this leads us to a contradiction. By Lemma 9.2, any multiple curve is contained in $\mathcal{E}_1 \cup \mathcal{E}_2$ because S_2 does not meet $\hat{F}$. Recall that the singular points of C^* are not infinitely near. Since $\tau^* T$ induces on each $\mathcal{E}_i$ a fiber of the natural map $\mathcal{E}_i \to \mathbf{P}^1$, we see that the possible multiple curve is Ξ defined by $u_2 = y_2 = 0$ on U_2 and $u_1 = x_2 = 0$ on V_2. This occurs when f_1 and f_3 vanish at $x = y = 0$ and f_2 vanishes twice at that point. Let $\tau_3 : W_3 \to W_2$ be the blowing-up with center Ξ, and let S_3 be the proper transform of S_2 by τ_3. We denote by $\mathcal{E}_3$ the exceptional divisor. Then S_3 is normal and $\omega_{S_3} = \mathcal{O}(\tau_3^* \tau^* T - \mathcal{E}_3)$. Thus $H^0(\omega_{S_3})$ is in bijection with $H^0(W, \mathcal{O}(T - \{x = y = 0\}))$. It follows $h^0(\omega_{S_3}) = p_g - 1$. Since the minimal resolution of S_3 is birational to S, this is impossible. Thus S_2 is normal.

Let $\hat{W}$ and S' be as in 9.4. Then $\hat{W}$ is obtained from W_2 by contracting the curve Ξ to the point ξ. If $f_2(0,0) \neq 0$, then S_2 does not meet Ξ. Thus $S' \simeq S_2$ has only RDP as the above calculation shows. On the other hand, if $f_2(0,0) = 0$, S_2 is obtained from S' by blowing up ξ. We show that ξ is a RDP on S'. Let $\sigma : X \to S_2$ be the minimal resolution. Since X is birationally equivalent to S, we have $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_S)$. Then we see that S_2 has only RDP by $\chi(\mathcal{O}_{S_2}) = \chi(\mathcal{O}_X)$. We in particular have $\omega_X = \sigma^* \omega_{S_2} = \mathcal{O}(\sigma^* \tau^* T)$. Let Ξ' be the proper transform of Ξ by σ. Then

$$
\omega_X \Xi' = (\sigma^* \omega_{S_2}) \Xi' = (\sigma^* \omega_{S_2})(\sigma^* \Xi) = \omega_{S_2} \Xi = 0.
$$

Thus Ξ' is a (-2)-curve on X. Then, since S_2 has only RDP, each irreducible component of $\sigma^* \Xi$ is a (-2)-curve. This implies that ξ is a RDP on S'. Thus S' has only RDP and S is the minimal resolution.

In order to describe the canonical degenerate fiber, we let $\lambda : \mathcal{E}_1 \to \mathcal{E}' \simeq \Sigma_1$ be the restriction of τ_2 to $\mathcal{E}_1$. This is the blowing-up with center $\mathcal{E}' \cap L_2'$ and its exceptional curve E can be identified with the intersection $\mathcal{E}_1 \cap \mathcal{E}_2$. With these notations, $S_2|_{\mathcal{E}_1}$ is linearly equivalent to $\lambda^*(2\Delta_0 + 4\Gamma) - 2E$. Thus it is the proper transform of a curve of arithmetic genus 2 with a double point. Since $S_2|_{\mathcal{E}_2} \sim 2\Delta_0 + 4\Gamma$ on $\mathcal{E}_2 \simeq \Sigma_2$, it is an elliptic curve.

Thus, in general, the canonical degenerate fiber consists of two elliptic curves meeting at two points. The self-intersection number of each curve is -2.

9.7 We suppose that S^* has the singular fiber of type 3). For any $P \in L$, we can take a small open neighbourhood U of P in W with coordinates $(t,\ x,\ y)$, where t is as in 9.4 and $(x,\ y)$ is an affine coordinate system on $F_s \simeq \mathbf{P}^2$ such that L is defined by $x = 0$. Then, on U, the equation (15) can be rewritten as

$$x^4 + f_1 x^2 t + \sum_{i=2}^{4c-(p_g-6)} f_i t^i = 0,$$

where the $f_i = f_i(x, y)$ are functions in x, y. Since L is a double curve, $f_2(0, y)$ is not identically zero.

Let $\nu : \hat{W} \to W$ and S' be as in 9.4. We denote by $\mathcal{E}'$ and F' the exceptional divisor and the proper transform of F_s, respectively. We cover $\nu^{-1}(U)$ by open sets U_1, U_2 with coordinates (u_1, v_1, y_1) and (u_2, v_2, y_2), respectively, where

$$t = u_1 v_1 = u_2,\ x = u_1 = u_2 v_2,\ y = y_1 = y_2.$$

Then S' is defined by

$$\begin{aligned} &u_1^2 + f_1 u_1 v_1 + \textstyle\sum_{i \geq 2} f_i u_1^{i-2} v_1^i = 0 \quad \text{on } U_1, \\ &u_2^2 v_2^4 + f_1 u_2 v_2^2 + \textstyle\sum_{i \geq 2} f_i u_2^{i-2} = 0 \quad \text{on } U_2. \end{aligned}$$

We let $\tilde{\nu} : \tilde{W} \to \hat{W}$ be the blowing-up with center $F' \cap \mathcal{E}'$ and put $\tau = \nu \circ \tilde{\nu}$. We denote by $\tilde{F}$, $\mathcal{E}_1$ and $\mathcal{E}_2$ the proper transforms of F', $\mathcal{E}'$ and the exceptional divisor of $\tilde{\nu}$, respectively. Then $\mathcal{E}_1 \simeq \Sigma_1$ and $\mathcal{E}_2 \simeq \Sigma_2$. We have $\mathcal{E}_2|_{\mathcal{E}_1} = \Delta_0$, $\mathcal{E}_1|_{\mathcal{E}_2} \sim \Delta_0 + 2\Gamma$ and $\tilde{F}|_{\mathcal{E}_2} = \Delta_0$. Further, we can show $\mathcal{E}_1|_{\mathcal{E}_1} \sim -2\Delta_0$ and $\mathcal{E}_2|_{\mathcal{E}_2} \sim -\Delta_0 - \Gamma$.

Let S'' be the proper transform of S^* by τ. Let V be a open neighbourhood of $Q \in F' \cap \mathcal{E}'$. We can assume (u_1, v_1, y_1) forms a system of local coordinates on V. We cover $\tilde{\nu}^{-1}(V)$ by two open sets V_1, V_2 with coordinates (w_1, x_1, y_1) and (w_2, x_2, y_1), respectively, which satisfies

$$u_1 = w_1 x_1 = w_2,\ v_1 = w_1 = w_2 x_2.$$

Then S'' is defined by

$$\begin{aligned} &x_1^2 + f_1 x_1 + f_2 + \textstyle\sum_{i \geq 3} f_i w_1^{2i-4} x_1^{i-2} = 0 \quad \text{on } V_1, \\ &1 + f_1 x_2 + f_2 x_2^2 + \textstyle\sum_{i \geq 3} f_i w_2^{i-1} x_2^i = 0 \quad \text{on } V_2. \end{aligned}$$

Thus S'' does not meet $\tilde{F}$. $S'' \cap \mathcal{E}_1$ is defined by $u_2 = f_2(0, y_2) = 0$ on U_2 and $x_1 = f_2(0, y_1) = 0$ on V_1. Further, $S'' \cap \mathcal{E}_2$ is defined by $w_1 = x_1^2 + f_1(0, y_1)x_1 + f_2(0, y_1) = 0$ on V_1 and $w_2 = 1 + f_1(0, y_1)x_2 + f_2(0, y_1)x_2^2 = 0$ on V_2. Thus $S'' \to S'$ is finite.

We have $S'' \sim \tau^*(4T-(p_g-6)F)-2\mathcal{E}_1-4\mathcal{E}_2$. Since $\tilde{F} \sim \tau^*F-\mathcal{E}_1-2\mathcal{E}_2$ and since it does not intersect with S'', we get $\omega_{S''} \sim \tau^*T|_{S''}$. Then we can calculate the invariants of S'' as in 9.5. We have $\omega_{S''}^2=3p_g-6$, $h^0(\omega_{S''})=p_g$ and $h^1(\omega_{S''})=0$. In particular, we have $\chi(\mathcal{O}_S)=\chi(\mathcal{O}_{S''})$. We can show that S'' has only isolated singular points as in 9.7. Thus S'' has only RDP. Further, we see that S'' is the normalization of S'. In particular, we have a holomorphic map $S \to S''$.

Since we have $S''|_{\mathcal{E}_1} \sim 4\Gamma$ and $S''|_{\mathcal{E}_2} \sim 2\Delta_0+4\Gamma$, the canonical degenerate fiber D_s consists of an elliptic curve R and 4 rational curves l_i, $1 \le i \le 4$, in general. Each l_i intersects with R at a point. Since $\mathcal{E}_1^2 S''=-8$ and $\mathcal{E}_2^2 S''=-2$, each of them has self-intersection number -2.

9.8 In order to show the existence of surfaces of type (s.3), we consider the surface S^* in $\mathbf{P}_{a,b,c}$ defined by

$$\alpha_0^2\mathcal{P}+\alpha_1\mathcal{Q}=0, \tag{16}$$

where

$$\begin{aligned}&\alpha_0 \in H^0(\mathbf{P}_{a,b,c},\mathcal{O}(F)),\\ &\alpha_1 \in H^0(\mathbf{P}_{a,b,c},\mathcal{O}(\epsilon F)),\\ &\mathcal{P} \in H^0(\mathbf{P}_{a,b,c},\mathcal{O}(4T-(p_g-4)F)),\\ &\mathcal{Q} \in H^0(\mathbf{P}_{a,b,c},\mathcal{O}(2T-((p_g-6+\epsilon)/2)F)),\end{aligned}$$

with $\epsilon=0$ or 1 according to p_g is even or odd. We assume that the α_i, $\mathcal{P}$ and $\mathcal{Q}$ are general. Let (X_0,X_1,X_2) be as in the proof of Lemma 9.3. We put $\mathcal{P}=\sum p_{ij}X_0^{4-i-j}X_1^iX_2^j$ and $\mathcal{Q}=\sum q_{ij}X_0^{2-i-j}X_1^iX_2^j$ as in (14).

By the argument found in [2, §2], we can show the following:

i) Consider the linear system $|4T-(p_g-4)F|$. If $2a+1 \ge c$, then its general member is irreducible and nonsingular. If $a+c \le 3b+1$ and $b \le 2a+1$, then its general member is irreducible and has only RDP.

ii) Consider the linear system $|2T-((p_g-6+\epsilon)/2)F|$. If $b+c \le 3a+2$, then it is free from base points. If $a+c \le 3b+3$, then its general member is irreducible and has at most singular points of type A_1.

In i) and ii), the singular points are on the intersection of the curve B defined by $X_1=X_2=0$ and the fibers defined by $p_{01}=0$ and $q_{01}=0$, respectively. Further, we remark that the following holds.

iii) If $a+c=3b+2$, $3b+3$, then X_2 is a fixed component of $|4T-(p_g-4)F|$. In this case, we can write $\mathcal{P}=X_2\mathcal{P}'$ with $\mathcal{P}' \in H^0(\mathcal{O}(3T-(p_g-4-c)F))$. The linear system $|3T-(p_g-4-c)F|$ is free from base points if $b \le 2a+1$. Thus we can assume that $(\mathcal{P})$ is a divisor with simple normal crossings.

We show that S^* has only RDP except for the double curve C_s defined by $\alpha_0=\mathcal{Q}=0$, assuming that $a+c \le 3b+3$ and $b \le 2a+1$. Let ξ be a singular point of $S^*\backslash C_s$. Then it is contained in the set of singular points of $(\mathcal{P})$ on $(\mathcal{Q})$.

We first consider the case $a+c \leq 3b+1$. Then, by i), $(\mathcal{P})$ has only isolated singular points on B. If $2a+1 \geq c$, then we can assume that $(\mathcal{P})$ is nonsingular. If $b+c \leq 3a+2$, then we can assume that $(\mathcal{Q})$ does not pass through singular points of $(\mathcal{P})$. Thus S^* is nonsingular except for C_s in these cases. So we can assume that $2a+1 < c$ and $b+c > 3a+2$. Then $(\mathcal{P})$ and $(\mathcal{Q})$ contain B. The singular point ξ is a zero of p_{01}. Put $x_1 = X_1/X_0$ and $x_2 = X_2/X_0$. Then, in a neighbourhood of B, (x_1, x_2) forms a system of affine coordinates on each fiber of π. We take a local parameter t of B at ξ. We can assume $p_{01} = t$ in a neighbourhood of ξ. Then we can rewrite (16) locally as

$$\begin{aligned} & -x_2(\alpha_0^2 t + \cdots) \\ = \ & (\alpha_0^2 p_{20} + \alpha_1 q_{10}^2)x_1^2 + (\alpha_0^2 p_{30} + 2\alpha_1 q_{10} q_{20})x_1^3 + (\alpha_0^2 p_{40} + \alpha_1 q_{20}^2)x_1^4 \end{aligned}$$

Thus ξ is a RDP of type A_n, for some $n \leq 3$, if we choose α_i, p_{ij}, q_{ij} general.

We next consider the case $a+c \geq 3b+2$. Let $\mathcal{P}'$ be as in iii). Since $b \leq 2a+1$, we can assume that $(\mathcal{P}')$ meets transversely with B. If $a = b$, then $Bs|2T-((p_g-6+\epsilon)/2)F| = \emptyset$ and we can assume that the divisor $(X_2)+(\mathcal{P}')+(\mathcal{Q})$ has only normal crossings. Thus ξ is a A_1-singularity as (16) shows. If $a < b$, then $q_{00} = q_{10} = 0$. Thus $(\mathcal{Q}) \cap (X_2) = B$ and $(\mathcal{Q})$ contacts (X_2) at a generic point of B. Thus $\xi \in B$. Then a local study as in the previous case shows that it is a RDP of S^*.

Remark 9.9 The singular fibers of type 3) are implicitly used in [1] to construct surfaces with pencils of nonhyperelliptic curves of genus 3. Note that we get a simple elliptic singularity of type $\tilde{E}_7$ if we contract the curve R in 9.7.

References

[1] T. Ashikaga: On the geography of surfaces with birational canonical morphisms, preprint (1989).

[2] T. Ashikaga and K. Konno: Algebraic surfaces of general type with $c_1^2 = 3p_g - 7$, preprint (submitted to Tôhoku Math. J.)

[3] A. Beauville: L'application canonique pour les surfaces de type général, Invent. Math. 55 (1979), 121–140.

[4] T. Fujita: On the structure of polarized varieties with Δ-genus zero, J. Fac. Sci. Univ. Tokyo 22 (1975), 103–115.

[5] T. Fujita: On the structure of polarized manifolds with total deficiency one, I, II and III, J. Math. Soc. Japan 32 (1980), 709–725; ibid. 33 (1981), 415–434; ibid. 36 (1984), 75–89.

[6] T. Fujita: On polarized varieties of small Δ-genera, Tôhoku Math. J. 34 (1982), 319–341.

[7] T. Fujita: Projective varieties of $\Delta-$genus one, in: *Algebraic and Topological Theories – to the memory of Dr. Takehiko MIYATA*, Kinokuniya Book Store (1985), 149–175.

[8] J. Harris: A bound on the geometric genus of projective varieties, Ann. Sc. Norm. Sup. Pisa Ser. IV vol. VIII (1981), 35–68.

[9] J. Harris: *Curves in Projective Space*, Le presses de l'Université de Montreal (1982).

[10] E. Horikawa: On deformations of quintic surfaces, Invent. Math. 31 (1975), 43–85.

[11] E. Horikawa: Algebraic surfaces of general type with small c_1^2, I, II, III, IV and V, Ann. of Math. 104 (1976), 358–387; Invent. Math. 37 (1976), 121–155; ibid. 47 (1978), 209–248; ibid. 50 (1979), 103–128; J. Fac. Sci. Univ. Tokyo 28 (1981), 745–755.

[12] M. Nagata: On rational surfaces I, Mem. Coll. Sci. Univ. Kyoto, Ser. A 32 (1960), 351–370.

[13] G. Xiao: Hyperelliptic surfaces of general type with $K^2 < 4\chi$, manuscripta math. 57 (1987), 125–148.

Deformations of a Complex Manifold near a Strongly Pseudo-Convex Real Hypersurface and a Realization of Kuranishi Family of Strongly Pseudo-Convex CR Structures

by

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Introduction

Let (V, o) be a normal isolated singularity . If we cut V with a small real sphere centered at o , we have a real hypersurface M of V with a natural strongly pseudo-convex CR structure . Deformations of strongly pseudo-convex CR structures on M with $\dim_{\mathbf{R}} M \geq 5$ were considered first by M.Kuranishi (cf. [Ku3]) in order to treat deformations of (V, o) differential-geometrically, and were improved later by T.Akahori for M with $\dim_{\mathbf{R}} M \geq 7$ (cf. [A1]) . It is a natural question to ask whether the Kuranishi family of strongly pseudo-convex CR structures on M obtained in [A1] (also cf. [M1]) actually induces the versal family of deformations of (V, o) (cf. [G]) . In the case of $\operatorname{codh}(V, o) \geq 3$, by [F] , the latter versal family is obtained as the maximal Stein completion of the versal family of deformations of its regular part $V \setminus o$.

On the other hand , realizations of deformations of abstract CR structures are one of basic interests of deformation theory of CR structures . Under some cohomological conditions , G.K.Kiremidjian has done such a realization as boundaries of deformations of the interior domain (cf. [Ki 1, 2, 3]) . But in order to approach the above problem , it is enough to consider embeddings of deformations of CR structures into deformations of a tubler neighbourhood of M .

So the main result of this paper is the following relative version of T.Ohsawa's embedding theorem of a strongly pseudo-convex CR structure (cf. [O]) :

Theorem 2. *Let M be a smooth strongly pseudo-convex compact real hypersurface of a complex manifold X with* $\dim_{\mathbb{C}} X \geq 4$. *Then the Kuraniahi family of strongly pseudo-convex CR structures on M is realized as a real hypersurface of the formally versal convergent family of deformations of X near M .*

Hence , as stated above , we have

Corollary. *Let (V, o) be a normal isolated singularity with* $\dim_{\mathbb{C}}(V, o) \geq 4$ *and* $\mathrm{codh}(V, o) \geq 3$. *Let $M = V \cap S$ with a sufficiently small real sphere S centered at o . Then the Kuranishi family of strongly pseudo-convex CR structures on M is realized as a real hypersurface of the versal family of deformations of (V, o) .*

The existence of the formally versal convergent family of deformations of X near M (Theorem 1) was essentially proved in [A2, 3] . We will prove Theorem 2 as an application of a formal deformation theory of X near M . That is , we will show that the versal map from the parameter space of the family in Theorem 1 into the one of the Kuranishi family of CR structures is formally isomorphic . The essential part of the proof is to construct the formal inverse map by showing that the Kuranishi family of CR structures on M is extendable to a formal family of complex structures on a tubler neighbourhood of M . This is a consequence of the isomorphisms $H^{i}_{\bar\partial}(\bar\Omega, T'X) \simeq H^{i}_{\bar\partial_b}(M, T'X_{|M})$ for $i = 1, 2$, which was obtained in [Y] , where Ω is a tubler neighbourhood of M in X .

The arrangement of this paper is as follows . In §1 , we will give a formulation of deformations of X near M . In §2 , we will adjust arguments in [A2], [A-K] and [Ku1, 2] to prove Theorem 1 . Though they treated only reduced deformations , we will consider non-reduced ones as well by slightly modifying their arguments . We will prove Theorem 2 in §3 .

In more general situations dealt with in [M2] , the method in this paper works well and Theorems 1 and 2 hold .

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1. Formulation of deformations of a complex manifold near a real hypersurface

Let X be a complex manifold with $\dim_{\mathbb{C}} X = N \geq 4$ and M a strongly pseudo-convex compact real hypersurface of X. We will consider a functor of deformations of X near M: $\mathcal{F}_M : (Gan) \to (Sets)$, where (Gan) denotes the category of germs of complex spaces, given by

$$\mathcal{F}_M((T,o)) = \{(\mathfrak{X},\pi) \mid \text{(i) a complex space } \mathfrak{X},$$
$$\text{(ii) a smooth morphism } \pi : \mathfrak{X} \to (T,o),$$
$$\text{(iii) } \pi^{-1}(o) \simeq \text{a neighbourhood of } M \text{ in } X\}/\sim,$$

where $(\mathfrak{X},\pi) \sim (\mathfrak{X}',\pi')$ if there exist neighbourhoods $\mathfrak{X}_1$ and $\mathfrak{X}_1'$ of $M \times o$ in $\mathfrak{X}$ and $\mathfrak{X}'$ respectively and a complex analytic isomorphism $\chi : \mathfrak{X}_1 \to \mathfrak{X}_1'$ such that $\pi' \circ \chi = \pi$.

We may assume that there exists a smooth strictly plurisubharmonic function $r : X \to (a_*, b^*)$, $-\infty \leq a_* < 0 < b^* \leq +\infty$ such that $dr \neq 0$ on X, $\Omega_{a,b} = \{a < r(x) < b\}$ is relatively compact in X for any $a_* < a < b < b^*$ and $M = \{r(x) = 0\}$.

Because $\mathcal{F}_M(\mathrm{Spec}\ \mathbb{C}[t]/(t^2)) = \varinjlim_{U \supset M} H^1(U, \Theta_U)$, and by [A-G], $H^1(\Omega_{a,b}, \Theta)$ measures the infinitesimal deformation space of $\mathcal{F}_M$ for any $a_* < a < 0 < b < b^*$, where Θ denotes the sheaf of germs of holomorphic vector fields on X.

Theorem 1. *Under the above assumption and if* $a_* < a < 0 < b < b^*$, *there exists a family* $\pi : \mathfrak{X} \to T$ *of complex manifolds with* $\pi^{-1}(o) \simeq \Omega_{a,b}$ *and such that it gives a formally versal convergent family for* $\mathcal{F}_{M_c}$ *for any* $a < c < b$. *Where we denote* $M_c = \{r(x) = c\}$.

We will prove Theorem 1 in §2 .

2. Proof of Theorem 1

We fix $a_*<a_1<a<0<b<b_1<b^*$ and an hermitian metric on X which is real analytic on a neighbourhood of $\bar{\Omega}_{a,b}$ and equals to the Levi metric near $\partial\Omega_{a_1,b_1}$. We denote $\Omega=\Omega_{a,b}$ and $\Omega_1=\Omega_{a_1,b_1}$.

In the first place , we will construct a versal family (in Kuranishi's sense) of complex structures of $\mathscr{A}^m$-class over $\bar{\Omega}_1$, by means of the method in [A2] . As shown in [A3] , we are able to apply the method even to a (1,1)-convex-concave domain Ω_1 .

Let $r=\dim_{\mathbb{C}}H^1(\bar{\Omega}_1, T'X)$ and denote $(t)=(t_1,\dots,t_r)\in\mathbb{C}^r$. Let $\| \ \|'_{(0,m)}$ be a norm , between the Sobolev $(m+1)$-norm and the tangential Sobolev $(0,m)$-norm , introduced in [A2] . In [A2] , we obtained a convergent powerseries $\varphi(t)\in\mathscr{A}^{0,1}_m\{t_1,\dots,t_r\}$, where $\mathscr{A}^{0,q}_m$ denotes the completion of $A^{0,q}(\bar{\Omega}_1, T'X)$ with respect to $\| \ \|'_{(0,m)}$-norm , such that

(2.1) $\varphi(0)=0$,

(2.2) if we denote by $\varphi_1(t)$ the linear term of $\varphi(t)$, then $\bar{\partial}\varphi_1(t)=0$ and $\varphi_1(t)$ spans $H^{0,1}(\bar{\Omega}_1, T'X)$,

(2.3) $\vartheta\varphi(t)=0$ on a neighbourhood of $\bar{\Omega}$,

(2.4) $\bar{\partial}\varphi(t)-(1/2)\bar{\partial}\vartheta N[\varphi(t),\varphi(t)]=0$.

Where N is the Neumann operator for a new boundary condition introduced in [A2] . We recall the properties of N which will be needed in our arguments (cf. [A2], Theorem 5.1) .

(2.5) $\psi = H\psi + (\bar{\partial}\vartheta+\vartheta\bar{\partial})N\psi$,

(2.6) If $\bar{\partial}\psi=0$ and $H\psi=0$ then $\psi=\bar{\partial}\vartheta N\psi$, where H is the projection onto the harmonic space H^2 with respect to the new boundary condition , and

(2.7) H^2 is naturally isomorphic to $H^{0,2}(\bar{\Omega}_1, T'X)$.

Let h be a complex analytic map from a neighbourhood W of $0\in\mathbb{C}^r$ into H^2 given by $h(t)=H[\varphi(t),\varphi(t)]$, and $T=h^{-1}(0)$.

Let $P(\varphi(t))=\bar{\partial}\varphi(t)-(1/2)[\varphi(t),\varphi(t)]$.

Proposition 2.1. *$P(\varphi(t))\in\mathcal{I}_{T,o}\otimes_{\mathbb{C}}\mathcal{A}^{0,2}_{(0,m)}\{t_1,\dots,t_r\}$, where $\mathcal{A}^{0,2}_{(0,m)}$ denotes the completion of $A^{0,2}(\bar{\Omega}_1, T'X)$ by tangential Sobolev $(0,m)$-norm .*

The following Lemma implies Proposition 2.1 by the Grauert Division Theorem (cf. [G]) .

Lemma 2.2. *$P(\varphi(t))\equiv 0 \mod (\mathfrak{m}^{\mu+1}+\mathcal{I}_{T,o})\otimes_{\mathbb{C}}\mathcal{A}^{0,2}_{(0,m)}\{t_1,\dots,t_m\}$ for all $\mu\geq 0$.*

Proof. We will prove Lemma 2.2 by the induction on μ . For $\mu=1$, it is clear because $P(\varphi(t))\in\mathfrak{m}_t^2\otimes_{\mathbb{C}}\mathcal{A}^{0,2}_{(0,m)}\{t\}$.

Assume that $P(\varphi(t))\equiv 0 \mod (\mathfrak{m}^{\mu+1}+\mathcal{I}_{T,o})\otimes_{\mathbb{C}}\mathcal{A}^{0,2}_{(0,m)}\{t\}$ for $\mu\geq 1$. By (2.4) , $P(\varphi(t)) =(1/2)\bar{\partial}\vartheta N[\varphi(t),\varphi(t)]-(1/2)[\varphi(t),\varphi(t)]$. If we denote by $\psi(t)$ the canonical modulo of $[\varphi(t),\varphi(t)]$ mod $(\mathfrak{m}^{\mu+2}+\mathcal{I}_{T,o})\otimes_{\mathbb{C}}\mathcal{A}^{0,2}_{(0,m)}\{t\}$ by the Grauert division , then we have

$$P(\varphi(t)) \equiv (1/2)\bar{\partial}\vartheta N\psi(t)-(1/2)\psi(t) \mod (\mathfrak{m}^{\mu+2}+\mathcal{I}_{T,o})\otimes_{\mathbb{C}}\mathcal{A}^{0,2}_{(0,m)}\{t\} .$$

Hence $H\psi(t)=0$. Since $\bar{\partial}[\varphi(t),\varphi(t)]=2[P(\varphi(t)),\varphi(t)]$, we have $\bar{\partial}\psi(t)=0$ by the inductive assumption . Therefore we have $\psi(t)=\bar{\partial}\vartheta N\psi(t)$ by (2.6) . Q.E.D.

By (2.3) and (2.4), $\varphi(t)$ satisfies an elliptic partial differential equation $\sum_{i,j}(\partial^2\varphi/\partial t_i\partial\bar{t}_j)+\square\varphi-(1/2)\vartheta[\varphi,\varphi]=0$ and also satisfies $\partial\varphi/\partial\bar{t}=0$, over a neighbourhood of $\bar{\Omega}\times D$, hence $\varphi(t)$ is real analytic and depends complex analytically on t on a neighbourhood of $\bar{\Omega}\times D$ where D is a neighbourhood of 0 in $\mathbb{C}^r$.

Proposition 2.3. *There exists a family of complex maniflods* $\pi:\mathfrak{X}\to T$ *with* $\pi^{-1}(o)\simeq\Omega$ *and with a real analytic isomorphism* $G:\Omega\times T\to\mathfrak{X}$ *such that* (i) $G_{|\Omega\times o}=id_{X|\Omega}$,
(ii) G *is a complex analytic map with respect to the complex structure on* $\Omega\times T$ *defined by* $\varphi(t)$,
(iii) G *is complex analytic on* $p\times T$ *for each fixed* $p\in\Omega$.

Proof. By ideal-theoretically improvement of the argument in the proof of Proposition 7.3 of [Ku2], we can obtain a collection of families of local charts. By the same argument in the proof of Proposition 1.3 of [Ku1] using the Grauert Division theorem instead of Lemma 1.3 of [Ku1], we have an analytic space $\mathfrak{X}$ by patching these local charts together. Q.E.D.

Next we will show that this family $\pi:\mathfrak{X}\to(T,o)$ is a formally versal family for $\mathcal{F}_{M_c}$ for any $a<c<b$.

Let $a<a_2<c<b_2<b$ and $\Omega_2=\Omega_{a_2,b_2}$. Then, by the same arguments in §3 of [A-K], slightly modified with the Grauert Division Theorem (cf. [G]) and relying on the isomorphism $H^i(\bar{\Omega}_1,T'X)\simeq H^i(\bar{\Omega}_2,T'X)$ for $i=1,2$ (cf. [H] Theorem 3.4.8), we have

Proposition 2.4. *Let (T,o) and $\varphi(t)$ be as constructed above. If $\pi':\mathfrak{X}' \to (S,o)$ be a family of complex manifolds with $\pi'^{-1}(o) \simeq$ a neighbourhood of $\bar{\Omega}_2$. Suppose a morphism $\alpha^\mu: S_\mu \to T_\mu$ and an embedding $g^\mu: \bar{\Omega}_2 \times S_\mu \to \mathfrak{X}' \times_S S_\mu$ are given for some $\mu \geq 0$ satisfying*

$(1)_\mu$ $p_2' \circ g^\mu = p_2$ and

$(2)_\mu$ $(\bar{\partial} - \varphi(\alpha^\mu(s)))(\xi' \circ g^\mu(z,s)) \in (m_S^{\mu+1} + \mathcal{I}_{S,o}) \otimes_{\mathbf{C}} A^{0,1}(U \cap \bar{\Omega}_2, T'X)[[s]]$ for any charts (U,z) of X and $(\mathfrak{U}',(\xi',s))$ of $\mathfrak{X}'$, where p_2' and p_2 denote projections onto the second factors. Then there exist extensions $\alpha^{\mu+1}: S_{\mu+1} \to T_{\mu+1}$ and $g^{\mu+1}: \bar{\Omega}_2 \times S_{\mu+1} \to \mathfrak{X}' \times_S S_{\mu+1}$ satisfying $(1)_{\mu+1}$ and $(2)_{\mu+1}$. Where by subscript μ we denote the infinitesimal neighbourhood of order μ.

We will now show that Proposition 2.4 implies the following formal versality of $\pi:\mathfrak{X} \to (T,o)$.

Proposition 2.5. *Let $\pi:\mathfrak{X} \to (T,o)$ be as in Proposition 2.3 and $\pi':\mathfrak{X}' \to (S,o)$ be a family of complex manifolds with $\pi'^{-1}(o) \simeq$ a neighbourhood of M_c. If a morphism $\alpha^\mu: S_\mu \to T_\mu$ and an isomorhism $H^\mu: \mathfrak{X}_1' \times_S S_\mu \to \mathfrak{X}_1 \times_T S_\mu$ are given for some $\mu \geq 0$, where $\mathfrak{X}_1'$ and $\mathfrak{X}_1$ are neighbourhoods of $M_c \times o$ in $\mathfrak{X}'$ and $\mathfrak{X}$ respectively, then we have extensions $\alpha^{\mu+1}: S_{\mu+1} \to T_{\mu+1}$ and $H^{\mu+1}: \mathfrak{X}_2' \times_S S_{\mu+1} \to \mathfrak{X}_2 \times_T S_{\mu+1}$. Where $\mathfrak{X}_2'$ and $\mathfrak{X}_2$ are smaller neighbourhoods of $M_c \times o$ in $\mathfrak{X}$ and $\mathfrak{X}'$ respectively.*

Proof. We may assume that $\bar{\Omega}_2 \times o \subset \mathfrak{X}_1$. Let $G^{\mu}=(H^{\mu})^{-1}$ and $g^{\mu}=G^{\mu}\circ(\alpha^{\mu})^{*}(G_{|\bar{\Omega}_2\times T})$, where $(\alpha^{\mu})^{*}G$ is the real analytic isomorphism $\Omega\times S_{\mu}\to\mathfrak{X}\times_{T}S_{\mu}$ induced from G in Proposition 2.3 via α^{μ} . Then , by Proposition 2.4 , we have extensions $\alpha^{\mu+1}:S_{\mu+1}\to T_{\mu+1}$ and $g^{\mu+1}:\bar{\Omega}_2\times S_{\mu+1}\to\mathfrak{X}'\times_{S}S_{\mu+1}$ satisfying $(1)_{\mu+1}$ and $(2)_{\mu+1}$ of Proposition 2.4 .

Let $\{\mathfrak{U}_i,(\xi_i,t)\}$, $\{\mathfrak{U}'_i,(\xi'_i,t)\}$ and $\{U_i,z_i\}$ be systems of local charts of $\mathfrak{X}$, $\mathfrak{X}'$ and X respectively and assume that $\xi_i=f_{ij}(\xi_j,t)$ on $\mathfrak{U}_i\cap\mathfrak{U}_j$, $\xi'_i=h_{ij}(\xi'_j,s)$ on $\mathfrak{U}_i\cap\mathfrak{U}_j$ and $z_i=\bar{f}_{ij}(z_j)$ on $U_i\cap U_j$. We denote $\xi_i\circ G(z_i,t)=\tilde{z}_i(z_i,t)$.

Suppose that the isomorphism G^{μ} is given by $\xi'_i=G^{\mu}_i(\xi_i,s)$. Let $\chi_{i|\mu+1}(s)$ be a homogeneous polynomial of s of degree $\mu+1$ given as the canonical modulo of $g^{\mu+1}_i(s)-G^{\mu}_i(\tilde{z}_i(z_i,\alpha^{\mu+1}(s)),s)$ mod $m_S^{\mu+2}+\mathcal{I}_{S,o}$ in $A^{0,0}(U_i\cap\bar{\Omega}_2,T'X)[[s]]$.

Lemma 2.6. $\bar{\partial}\chi_{i|\mu+1}(s)=0$.

Proof. $\bar{\partial}\chi_{i|\mu+1}(s)\equiv$
$(\bar{\partial}-\varphi(\alpha^{\mu+1}(s)))g^{\mu+1}_i(s)-(\partial G^{\mu}_i/\partial\xi_i)(\bar{\partial}-\varphi(\alpha^{\mu+1}(s))\tilde{z}_i(z_i,\alpha^{\mu+1}(s))$
$+\varphi(\alpha^{\mu+1}(s))\chi_{i|\mu+1}(s)\equiv 0 \mod m_S^{\mu+2}+\mathcal{I}_{S,o}$. Q.E.D.

Let $G_{i|\mu+1}(s)=\chi_{i|\mu+1}(s)$ and $G^{\mu+1}_i(s)=G^{\mu}_i(s)+G_{i|\mu+1}(s)$.

Lemma 2.7. *$G^{\mu+1}=\{G^{\mu+1}_i(s)\}$ is a holomorphic map from $\mathfrak{X}_2\times_T S_{\mu+1}$ into $\mathfrak{X}'_1\times_S S_{\mu+1}$* where $\mathfrak{X}_2$ is a smaller neighbourhood of $M_c\times o$.

Proof. Let $\Xi_{ij}^{\mu+1}(s)$ be the canonical modulo of $G_i^{\mu+1}(f_{ij}(\xi_j, \alpha^{\mu+1}(s)), s) - h_{ij}(G_j^{\mu+1}(\xi_j, s), s)$ mod $\mathfrak{m}_S^{\mu+2+\mathcal{J}}{}_{S,o}$ in $C^0(U_i \cap U_j \cap \Omega_2, \Theta)[[s]]$.

Then $\Xi_{ij}^{\mu+1}(\tilde{z}_j(z_j, \alpha^{\mu+1}(s)), s) \equiv g_i^{\mu+1}(z_j, s) - h_{ij}(g_j^{\mu+1}(z_j, s), s) \equiv 0$ mod $\mathfrak{m}_S^{\mu+2+\mathcal{J}}{}_{S,o}$.

Let $D = \{ord(h) \mid h \in \mathfrak{m}_S^{\mu+1+\mathcal{J}}{}_{S,o}\}$. Then ,

since $\Xi_{ij}^{\mu+1}(\xi_j, s) = \sum_{\nu \notin D} \xi_{ij}^{(\nu)}(\xi_j) s^\nu$ with $\xi_{ij}^{(\nu)}(z_j) \in C^1(\{U_i \cap \Omega_2\}, \Theta)$ and

$$\sum_{\nu \notin D} \xi_{ij}^{(\nu)}(z_j + o(s)) s^\nu \equiv 0 \text{ mod } \mathfrak{m}_S^{\mu+2+\mathcal{J}}{}_{S,o} ,$$

we have $\xi_{ij}^{(\nu)}(z_j) = 0$ by induction on $|\nu|$, where $o(s)$ is a term of order greater than zero with respect to s .

Hence $\Xi_{ij}^{\mu+1}(\xi_j, s) \equiv 0$ mod $\mathfrak{m}_S^{\mu+2+\mathcal{J}}{}_{S,o}$. Q.E.D.

Hence we have an extension $G^{\mu+1}$ of G^μ . It is clear that $(G^{\mu+1})^{-1}$ is an extension of H^μ .

Therefore Proposition 2.5 and Theorem 1 are proved .

3. Proof of Theorem 2

Let $\pi:\mathfrak{X} \to (T,o)$ be a family of complex manifolds with $\pi^{-1}(o)\simeq\Omega$ in §2 and $(\omega(s), (S,o))$ the Kuranishi family of CR structures on M with $\omega(o)=0$ obtained in [A1] or [M1] . Though we treated in [M1] only reduced parameters , we can improve the argument by the same way as in the proof of Proposition 2.1 .

By the same argument in [A-M] , improved with Grauert Division Theorem , we have a holomorphic map $\beta:(T,o) \to (S,o)$ and a C^k-embedding $g:M\times(T,o) \to \mathfrak{X}$.

Hence Theorem 2 will be reduced to the following Proposition .

Proposition 3.1. *β is an isomorphism* .

Proof. In order to prove this proposition , it is sufficient to show that β induces a formal isomorphism .

On M , we have a direct sum decomposition as C^∞-vector bundles ; $\mathbb{C}TM=\mathbb{C}F+{}^\circ T'+{}^\circ T''$ where ${}^\circ T'=T'X_{|M}\cap\mathbb{C}TM$, ${}^\circ T''=T''X_{|M}\cap\mathbb{C}TM$ and F is a real line bundle . We denote $\mathbb{C}F+{}^\circ T'$ by T' and $(\rho'_{|T'})^{-1}$ by τ where $\rho':\mathbb{C}TX_{|M}\to T'X_{|M}$ is the projection .

We denote by the same symbol τ the composition of the restriction map $A^{0,q}(\bar\Omega_{a,b},T'X) \to A_b^{0,q}(M,T'X_{|M})$ and the map $A_b^{0,q}(M,T'X_{|M}) \to A_b^{0,q}(M,T')$ induced from the above bundle isomorphism τ , where $a_*<a<0<b<b^*$ (cf. [Ak2] p. 319) .

Lemma 3.2. For any $a_*<a<0<b<b^*$, τ induces an isomorphism

$$H^{0,q}(\bar{\Omega}_{a,b}, T'X) \simeq H^q_{\bar{\partial}_b}(M, T') \quad (1\leq q\leq n-2) .$$

Proof. By the same arguments in pp. 81 and 82 of [Y] , we have

$$H^{0,q}(\bar{\Omega}_{a,0}, T'X) \simeq H^q_{\bar{\partial}_b}(M, T'X_{|M}) \quad (1\leq q\leq n-2) .$$

Since the bundle isomorphism $\tau:T'X_{|M}\to T'$ induces the isomorphisms of cohomologies $H^{0,q}_{\bar{\partial}_b}(M,T'X_{|M})\simeq H^{0,q}_{\bar{\partial}_b}(M,T')$, we infer the lemma from Theorem 3.4.8 of [H] . Q.E.D.

We note that Lemma 3.2 implies that $\beta_{|T_1}$ is an isomorphism .

Suppose that $\beta^{\mu}=\beta_{|T_{\mu}}:T_{\mu} \to S_{\mu}$ is an isomorphism for $\mu\geq 1$. We recall that $g(t)$ is represented by a formal power series $g_i(t)\in A^{0,0}(U_i\cap M, T'X)[[t]]$ for each local charts $(U_i,(z_i))$ of X and $(\mathfrak{U}_i,(\xi_i,t))$ of $\mathfrak{X}$ respectively (cf. [A-M] §3) . Let g^{μ} be the embedding $M\times T_{\mu} \to \mathfrak{X}\times_T T_{\mu}$ induced from g .

Since the sheaf of differentiable sections of $T'X$ vanishing on M is a fine sheaf , we can extend g^{μ} to an embedding $\tilde{g}^{\mu}$:(a neighbourhood of $\bar{\Omega}_3=\bar{\Omega}_{a_3,b_3}$)$\times T_{\mu} \to \mathfrak{X}\times_T T_{\mu}$ for some $a<a_3<0<b_3<b$. Let $\theta^{\mu}(t)$ be a family of complex structures over the neighbourhood of $\bar{\Omega}_3$ induced from the complex structure of $\mathfrak{X}$ via the embedding $\tilde{g}^{\mu}$. Let $\varphi^{\mu}(s)$ be a $A^{0,1}(\bar{\Omega}_3,T'X)$-valued polynomial of s of degree μ given by $\varphi^{\mu}(s)\equiv\theta^{\mu}((\beta^{\mu})^{-1}(s))$ mod $\mathfrak{m}_S^{\mu+1}$. Then we have

$(3.1)_{\mu}$ $P(\varphi^{\mu}(s))\equiv 0$ mod $\mathfrak{m}_S^{\mu+1}+\mathcal{J}_{S,o}$ in $A^{0,2}(\bar{\Omega}_3,T'X)[[s]]$, and

$(3.2)_{\mu}$ $\tau\varphi^{\mu}(s)\equiv\omega(s)$ mod $\mathfrak{m}_S^{\mu+1}+\mathcal{J}_{S,o}$ in $A_b^{0,1}(M,T')[[s]]$.

Let E_q be a subbundle of $T'\otimes\Lambda^q({}^\circ T'')^*$ such that $\Gamma(M,E_q)=\{\omega\in A_b^{0,q}(M,{}^\circ T')\mid \bar\partial_b\omega\in A_b^{0,q+1}(M,{}^\circ T')\}$ and $\mathcal{E}_q=\{\varphi\in A^{0,q}(\bar\Omega_3,T'X)\mid \tau\varphi\in\Gamma(M,E_q)\}$ (cf. [Ak1] Proposition 2.1 and [Ak2] p.323 respectively) .

Sublemma 1. *Let* $\varphi\in A^{0,1}(\bar\Omega_3,T'X)$ *and* ω *an almost CR structure on* M *induced from the almost complex structure* φ *on* $\bar\Omega_3$ *. Then* $\tau\varphi\in A_b^{0,1}({}^\circ T')$ *if and only if* $\omega\in A_b^{0,1}({}^\circ T')$ *. Moreover* $\tau\varphi=\omega$ *.*

Proof. Since ${}^\omega T''\subset{}^\varphi T''X_{|M}$, $u-\omega(u)\in{}^\varphi T''X_{|M}$ for $u\in{}^\circ T''$. If we denote by ρ' and ρ'' the projections from $\mathbb{C}TX_{|M}$ onto $T'X_{|M}$ and $T''X_{|M}$ respectively , $\varphi(u-\rho''\omega(u))=\rho'\omega(u)$. Since $\tau\rho'_{|T'}=id$, $\tau\varphi\in A_b^{0,1}({}^\circ T')$ if and only if $\omega\in A_b^{0,1}({}^\circ T')$. If $\omega\in A_b^{0,1}({}^\circ T')$, then $\varphi(u)=\omega(u)$ for $u\in{}^\circ T''$. This implies $\tau\varphi=\omega$. Q.E.D.

Let $P(\varphi)$ and $P_b(\omega)$ denote the integrability conditions of an almost complex structure φ on $\bar\Omega_3$ and of an almost CR structure ω on M respectively (cf. [A2] Definition 2.1 and [M1] (1.2) respectively) .

In the followings , by a subscript $T'X$ (resp. T' , ${}^\circ T''$ and $\mathbb{C}F$) we denote the projection onto $T'X$ (resp. T' , ${}^\circ T''$ and $\mathbb{C}F$) .

Sublemma 2. *If* $\varphi\in\mathcal{E}_1$ *, then* $\tau P(\varphi)=P_b(\tau\varphi)$ *.*

Proof. If $\tau\varphi\in\Gamma(M, E_1)$ we have

$\tau P(\varphi)(u,v) =(\tau\bar{\partial}\varphi)(u,v)+\tau([\varphi(u),\varphi(v)]_{T'X})-\tau\varphi([u,\varphi(v)]_{T''X}+[\varphi(u),v]_{T''X})$ for u, $v\in\Gamma(M, {}^{\circ}T'')$. (Cf. [Ak2] Definition 2.1.) Since $\varphi(u)$ and $\varphi(v)$ are in $\Gamma(M, {}^{\circ}T')$ we have $\tau([\varphi(u),\varphi(v)]_{T'X})=[\varphi(u),\varphi(v)]$. On the other hand $P_b(\tau\varphi)(u,v)=\bar{\partial}_b\tau\varphi(u,v)+[\tau\varphi(u),\tau\varphi(v)]_{T'}$ $-\tau\varphi([u,\tau\varphi(v)]_{{}^{\circ}T''}+[\tau\varphi(u),v]_{{}^{\circ}T''})-\tau\varphi([\tau\varphi(u),\tau\varphi(v)]_{{}^{\circ}T''})$ for u, v $\in\Gamma(M, {}^{\circ}T'')$. Since $\varphi(u)$ and $\varphi(v)$ are in $\Gamma(M, {}^{\circ}T')$ we have $P_b(\tau\varphi)(u,v)=\bar{\partial}_b\tau\varphi(u,v)+[\varphi(u),\varphi(v)]-\varphi([u,\varphi(v)]_{{}^{\circ}T''}+[\varphi(u),v]_{{}^{\circ}T''})$. Now, since $\tau\varphi\in A_b^{0,1}({}^{\circ}T')$ we have $\bar{\partial}_b\tau\varphi(u,v)= [\varphi(u),v]_{T'}+[u,\varphi(v)]_{T'}$ $-\varphi([u,v])$. Since $\bar{\partial}_b\tau\varphi(u,v)\in\Gamma(M, {}^{\circ}T')$, $[\varphi(u),v]_{\mathbb{C}F}+[u,\varphi(v)]_{\mathbb{C}F}=0$. Hence $[\varphi(u),v]_{T''X}+[u,\varphi(v)]_{T''X}=[\varphi(u),v]_{{}^{\circ}T''}+[u,\varphi(v)]_{{}^{\circ}T''}$. Thus we have $\tau P(\varphi)(u,v)=P_b(\tau\varphi)(u,v)$ for u, $v\in\Gamma(M, {}^{\circ}T'')$. Q.E.D.

As was shown in the proof of Theorem 2.4 of [A1], for any $\omega\in A_b^{0,q}(\mathbb{C}F)$, we have a $\theta\in A_b^{0,q-1}({}^{\circ}T')$ such that $\omega=(\bar{\partial}_b\theta)_{\mathbb{C}F}$. Hence we have

Sublemma 3. *Let B' be a finite dimensional subspace of $A_b^{0,q}(\mathbb{C}F)$. Then we have linear maps $\theta:B'\to A_b^{0,q-1}({}^{\circ}T')$ and $\tilde{\theta}:B'\to\mathcal{F}_{q-1}=\{\varphi\in A^{0,q-1}(\bar{\Omega}_3, T'X)\,|\,\tau(\varphi)\in A_b^{0,q-1}({}^{\circ}T')\}$ satisfying* (1) $\omega=(\bar{\partial}_b\theta(\omega))_{\mathbb{C}F}$ *and* (2) $\tau\tilde{\theta}=\theta$.

Sublemma 4. *Let B be a finite dimensional subspace of $A^{0,q}(\bar{\Omega}_3, T'X)$. Then there exist linear maps $p:B\to\mathcal{E}_q$ and $p_b:\tau(B)\to\Gamma(M, E_q)$ satisfying (1) $p_{|B\cap\mathcal{E}_q}=id$,* (2) *$p_{b|\tau(B)\cap\Gamma(M,E_q)}=id$ and* (3) $\tau p=p_b\tau$.

Proof. Let $B'_q=(\tau B)_{CF}$ and $B'_{q+1}=(\bar\partial_b\tau B)_{CF}$. Then , by sublemma 3 , we have linear maps $\tilde\theta_i : B'_i \to \mathcal{F}_{i-1}$ and $\theta_i : B'_i \to A_b^{0,i-1}({}^\circ T')$ satisfying (1) and (2) of sublemma 3 ($i=q$, $q+1$) . If we set $p(\varphi)=\varphi-\bar\partial\tilde\theta_q((\tau\varphi)_{CF})-\tilde\theta_{q+1}((\bar\partial_b\tau\varphi)_{CF})$ and $p_b(\omega)=\omega-\bar\partial_b\theta_q((\omega)_{CF})-\theta_{q+1}((\bar\partial_b\omega)_{CF})$, then it is clear that $p(\varphi)\in\mathcal{E}_q$, $p_b(\omega)\in\Gamma(M,E_q)$ and satisfy (1), (2) and (3) . Q.E.D.

Let $D=\{ord(h)\,|\,h\in\mathcal{I}_{S,o}\}$ and Λ be the reducing system of D . Then we have a system of generators $\{h_\alpha\}_{\alpha\in\Lambda}$ of $\mathcal{I}_{S,o}$ with the form $h_\alpha(s)=s^\alpha+g_\alpha$ $ord(g_\alpha)>\alpha$. By the Grauert Division Theorem (cf. [G]) , for any $f(s)\in A[[s]]$ there exists a unique $r=\sum_{\nu\notin D}r_\nu s^\nu\in A[[s]]$ such that $f(s)-r(s)\in\mathcal{I}_{S,o}\otimes A[[s]]$, where A denotes a $\mathbb{C}$-module .

By $(3.2)_\mu$, we have $\tau\varphi^\mu(s)-\omega(s)=\sum_{\alpha\in\Lambda}a_\alpha(s)h_\alpha(s)+\sum_{|I|\geq\mu+1,\, I\notin D}\beta_I s^I$ where $a_\alpha(s)\in A_b^{0,1}(T')[s]$ and $\beta_I\in A_b^{0,1}(T')$. Let B be a finite dimensional subspace of $A^{0,1}(\bar\Omega_3, T'X)$ which contains all coefficients of $\varphi^\mu(s)$ and such that $\tau(B)$ is generated by the coefficients of $\tau\varphi^\mu(s)$, $\omega(s)$, $a_\alpha(s)$ of degree less than $\mu+2$ and β_I for $|I|=\mu+1$. By sublemma 4 , we have linear maps $p:B\to\mathcal{E}_1$ and $p_b:\tau(B)\to\Gamma(M,E_1)$ satisfying (1), (2) and (3) of sublemma 4 .

If we set $\psi^\mu(s)=p(\varphi^\mu(s))$, then $\psi^\mu(s)\in\mathcal{E}_1[s]$ and satisfies

$(3.2)'_\mu$ $\tau\psi^\mu(s)-\omega(s)-\sum_{\alpha\in\Lambda}a'_\alpha(s)h_\alpha(s)-\sum_{|I|=\mu+1,\, I\notin D}\beta'_I s^I$

$\in m_S^{\mu+2}\otimes A_b^{0,1}(T')[[s]]$, where $a'_\alpha(s)\in\Gamma(M,E_1)[s]$ and $\beta'_I\in\Gamma(M,E_1)$.

Also by $(3.2)_\mu$, we have $(\tau\varphi^\mu(s))_{\mathbb{C}F}\in(\mathfrak{m}_S^{\mu+1}+\mathcal{J}_{S,o})\otimes A_b^{0,1}(\mathbb{C}F)[[s]]$ and $(\bar\partial_b\tau\varphi^\mu(s))_{\mathbb{C}F}\in(\mathfrak{m}_S^{\mu+1}+\mathcal{J}_{S,o})\otimes A_b^{0,2}(\mathbb{C}F)[[s]]$. Hence by the definition of p and p_b, we have $\psi^\mu(s)-\varphi^\mu(s)\in(\mathfrak{m}_S^{\mu+1}+\mathcal{J}_{S,o})\otimes A^{0,1}(\bar\Omega_3,T'X)[[s]]$. Hence by $(3.1)_\mu$, we have

$$P(\psi^\mu(s))\equiv P(\varphi^\mu(s))\equiv 0 \mod (\mathfrak{m}_S^{\mu+1}+\mathcal{J}_{S,o})\otimes A^{0,2}(\bar\Omega_3,T'X)[[s]] .$$

Since $P(\psi^\mu(s))\in\mathcal{E}_2[s]$, by the same argument as above, we have

$$(3.1)'_\mu \quad P(\psi^\mu(s))=\textstyle\sum_{\alpha\in\Lambda}b'_\alpha(s)h_\alpha(s)+\sum_{|I|=\mu+1,\, I\notin D}\gamma'_I s^I$$
$$\in\mathfrak{m}_S^{\mu+2}\otimes A^{0,2}(\bar\Omega_3,T'X)[[s]] ,\ \text{where } b'_\alpha(s)\in\mathcal{E}_2[s] \text{ and } \gamma'_I\in\mathcal{E}_2 .$$

Lemma 3.3. *There exists an extension* $\psi^{\mu+1}(s)$ *of* $\psi^\mu(s)$ *satisfying* $(3.1)'_{\mu+1}$ *and* $(3.2)'_{\mu+1}$.

Proof. Let $\psi'_{\mu+1}(s)$ be an $\mathcal{E}_1$-valued homogeneous polynomial of s of degree $\mu+1$ such that $\tau\psi'_{\mu+1}(s)$ is the canonical modulo of $\tau\psi^\mu(s)-\omega(s) \mod \mathfrak{m}_S^{\mu+2}+\mathcal{J}_{S,o}$. Then by $(3.2)'_\mu$ we have $\psi'_{\mu+1}(s)\in\mathcal{E}_1[s]$.

Let $p'_{\mu+1}(s)$ be the canonical modulo of $P(\psi^\mu(s)+\psi'_{\mu+1}(s))$ mod $\mathfrak{m}_S^{\mu+2}+\mathcal{J}_{S,o}$. Then, by $(3.1)'_\mu$ and since (the canonical modulo of $P(\psi^\mu(s)+\psi'_{\mu+1}(s))$ mod $\mathfrak{m}_S^{\mu+2}+\mathcal{J}_{S,o}$) = (the canonical modulo of $P(\psi^\mu(s))+\bar\partial\psi'_{\mu+1}(s)$ mod $\mathfrak{m}_S^{\mu+2}+\mathcal{J}_{S,o}$) = (The canonical modulo of $P(\psi^\mu(s))$ mod $\mathfrak{m}_S^{\mu+2}+\mathcal{J}_{S,o})+\bar\partial\psi'_{\mu+1}(s)$, we have $p'_{\mu+1}\in\mathcal{E}_2[s]$.

Since $\bar{\partial}P(\psi^{\mu}(s)+\psi'_{\mu+1}(s))=-[P(\psi^{\mu}(s)+\psi'_{\mu+1}(s)),\psi^{\mu}(s)+\psi'_{\mu+1}(s)]\equiv 0$ mod $m_S^{\mu+2}+\mathcal{I}_{S,o}$ in $A^{0,3}(\bar{\Omega}_3,T'X)[[s]]$, we have

(3.3) $\bar{\partial}p'_{\mu+1}(s)=0$.

By Sublemma 2 and the definition of $\psi'_{\mu+1}(s)$, we have

$\tau P(\psi^{\mu}(s)+\psi'_{\mu+1}(s))\equiv P_b(\omega(s))$ mod $m_S^{\mu+2}+\mathcal{I}_{S,o}$ in $\Gamma(M,E_2)[[s]]$.

Hence we have

(3.4) $\tau p'_{\mu+1}(s)=0$,

Now we consider a subcomplex $(\mathscr{C}^{0,\circ},\bar{\partial})$ of $T'X$-valued $\bar{\partial}$-complex . Let $\mathscr{C}^{0,q}=\{\varphi\in A^{0,q}(\bar{\Omega}_3,T'X)\mid \tau\varphi=0$ on $M\}$.

Then Lemma 3.2 implies

(3.5) $H^q_{\bar{\partial}}(\mathscr{C}^{0,\cdot})=0$ for $2\leq q\leq n-2$.

Hence , by (3.3) , (3.4) and (3.5) , there exists $\psi''_{\mu+1}(s)\in\mathscr{C}^{0,1}$ such that $\bar{\partial}\psi''_{\mu+1}(s)=-p'_{\mu+1}(s)$.

Hence , if we set $\psi^{\mu+1}(s)=\psi^{\mu}(s)+\psi'_{\mu+1}(s)+\psi''_{\mu+1}(s)$ then $\psi^{\mu+1}(s)$ satisfies $(3.2)'_{\mu+1}$ and

(3.6) $P(\psi^{\mu+1}(s))\equiv p'_{\mu+1}(s)+\bar{\partial}\psi''_{\mu+1}(s)=0$ mod $m_S^{\mu+2}+\mathcal{I}_{S,o}$ in $\mathcal{E}_2[[s]]$.

Hence we infer $(3.1)'_{\mu+1}$ from (3.6) . Q.E.D.

Lemma 3.4. *There exists a family of complex manifolds* $\pi':\mathfrak{X}'\to S_{\mu+1}$ *associated with the family* $\psi^{\mu+1}(s)$ *of complex structures and such that* $\mathfrak{X}'\times_{S_{\mu+1}}S_{\mu}\simeq\mathfrak{X}_1\times_{T_{\mu}}S_{\mu}$ *where* $\mathfrak{X}_1$ *is a neighbourhood of* $M\times o$ *in* $\mathfrak{X}$.

Proof. First, we will show the existence of families of complex charts. Let $\{V_\nu\}$ be a refinement of $\{U_i\}$ with $M\subset\bigcup_\nu V_\nu\subset\Omega_3$ and such that V_ν is holomorphically equivalent to a polydisc in a complex Euclidean space. Let $z_i^\mu(s)$ be the polynomial part of $\tilde{g}_i^\mu((\beta^\mu)^{-1}(s))$ of degree less than or equal to μ. Since $\varphi^\mu(s)\equiv\psi^\mu(s)$ mod $m_S^{\mu+1}+\mathcal{I}_{S,o}$ in $A^{0,1}(\bar{\Omega}_3,T'X)[[s]]$ we have

$(3.7)_\mu$ $(\bar\partial-\psi^\mu(s))z_i^\mu(s)\equiv 0$ mod $m_S^{\mu+1}+\mathcal{I}_{S,o}$ in $A^{0,1}(U_i,T'X)[[s]]$.

Let $\xi_{\mu+1}(s)$ be the canonical modulo of $-(\bar\partial-\psi^{\mu+1}(s))z_i^\mu(s)$ mod $m_S^{\mu+2}+\mathcal{I}_{S,o}$ in $A^{0,1}(U_i,T'X)[[s]]$. Then $\xi_{\mu+1}(s)$ is a homogeneous polynomial of s of degree $\mu+1$, and $\bar\partial\xi_{\mu+1}(s)=0$ because $\bar\partial(\bar\partial-\psi^{\mu+1}(s))z_i^\mu(s)\equiv P(\psi^{\mu+1}(s))z_i^\mu(s)$ mod $m_S^{\mu+2}+\mathcal{I}_{S,o}$ by $(3.7)_\mu$.

Since V_ν is a polydisc, there exists $z_{\nu|\mu+1}(s)$ such that $\bar\partial z_{\nu|\mu+1}(s)=\xi_{\mu+1}(s)|V_\nu$.

If we set $z_\nu^{\mu+1}(s)=z_{i(\nu)}^\mu(s)+z_{\nu|\mu+1}(s)$ then we have $(3.7)_{\mu+1}$.

Next, We obtain a family $\pi':\mathfrak{X}'\to S_{\mu+1}$ associated with $\psi^{\mu+1}(s)$ by the same argument as in the proof of Proposition 2.5. $\mathfrak{X}'\times_{S_{\mu+1}}S_\mu\simeq\mathfrak{X}_1\times_{T_\mu}S_\mu$ follows from $\psi^\mu(s)\equiv\varphi^\mu(s)$ mod $m_S^{\mu+1}+\mathcal{I}_{S,o}$ in $A^{0,1}(\bar\Omega_3,T'X)[[s]]$. Q.E.D.

By the formal versality of $\pi:\mathfrak{X}\to(T,o)$ for $\mathcal{F}_M$, we have an extension $\gamma^{\mu+1}:S_{\mu+1}\to T_{\mu+1}$ of $\gamma^\mu=(\beta^\mu)^{-1}$. Since $\gamma^{\mu+1}\circ\beta^{\mu+1}\equiv\gamma^1\circ\beta^1\equiv 1$ mod m_t^2 and $\beta^{\mu+1}\circ\gamma^{\mu+1}\equiv\beta^1\circ\gamma^1\equiv 1$ mod m_S^2, $\beta^{\mu+1}$ is isomorphic.

This completes the proofs of Proposition 3.1 and of Theorem 2.

References

[A1] Akahori, T.: The new estimate for the subbundles E_j and its application to the deformation of the boundaries of strongly pseudo convex domains. Inv. math. **63**, 311-334 (1981)

[A2] ---- : The new Neumann operator associated with deformations of strongly pseudo convex domains and its application to deformation theory. Inv. math. **68**, 317-352 (1982)

[A3] ---- : A criterion for the Neumann type problem over a differential complex on a strongly pseudo convex domain. Math. Ann. **264**, 525-535 (1983)

[A-M] --- and Miyajima, K.: Complex analytic construction of the Kuranishi family on a normal strongly pseudo convex manifold II. Publ. RIMS, Kyoto Univ. **16**, 811-834 (1980)

[A-G] Andreotti, A. and Grauert, H.: Théorèmes de finitude pour la cohomologie des espaces complexes. Bull. Soc. Math. France **90**, 193-259 (1962)

[F] Fujiki, A.: Flat Stein completion of a flat (1,1)-convex-concave map. Preprint

[G] Grauert, H.: Über die Deformation isolierter Singularitäten analytischer Mengen. Inv. math. **15**, 171-198 (1972)

[H] Hörmander, L.: L^2 estimates and existence theorems for the $\bar{\partial}$ operator. Acta Math. J. **113**, 89-152 (1965)

[Ki1] Kiremidjian, G.K.: Extendible pseudocomplex structures I. J. Approximation Theo. **19**, 281-303 (1977)

[Ki2] ----- : Extendible pseudocomplex structures II. J. Analyse Math. **30**, 304-329 (1976)

[Ki3] ---- : A direct extension method for CR structures. Math. Ann. **242**, 1-19 (1979)

[Ku1] Kuranishi,M.: On the locally complete families of complex analytic structures. Ann. of Math. **75**, 536-577 (1962)

[Ku2] ---- : *Deformations of compact complex manifolds.* Les Press Univ. Montreal, 1971

[Ku3] ---- : Deformations of isolated singularities and $\bar{\partial}_b^{\varphi}$. Preprint, Columbia Univ., 1972

[M1] Miyajima,K.: Completion of Akahori's construction of the versal family of strongly pseudo-convex CR structures. Trans. AMS **277**, 163-172 (1983)

[M2] ---- : Subellipticity of a boundary condition associated with construction of a versal family of complex structures on a domain with non-degenerate boundary. Math. Göttingensis **26**, 1988

[O] Ohsawa,T.: Global realization of strongly pseudoconvex CR manifolds. Publ. RIMS, Kyoto Univ. **20**, 599-605 (1984)

[Y] Yau,S.S.T.: Kohn-Rossi cohomology and its application to the complex Plateau problem, I. Ann. Math. **113**, 67-110 (1981)

Cohomological Criterion of Numerically Effectivity

by

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Let X be a d-dimensional non-singular projective variety and L a divisor on X. Then, if L is nef and big, by Kodaira-Kawamata-Viehweg vanishing Theorem, $H^i(X, \mathcal{O}_X(L+K_X)) = 0$ for all $i > 0$. In this note, we consider an inverse problem of the above Theorem.

We fix our situation. Let X and L be as above. We assume that L is big and there exists a very ample divisor A such that for any general member $B \in |A|$, $L_{|B}$ is nef on B. Then, we have

Theorem If $\limsup_{m\to\infty} \dfrac{\dim H^1(X, \mathcal{O}_X(K_X + mL))}{m^d} = 0$, then L is nef.

We start the following claim.

Claim 1. $H^i(X, \mathcal{O}_X(K_X + mL)) = 0$, for $m > 0$, $i > 1$.

We take a general member B of $|A|$ such that $L_{|B}$ is nef. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(K_X + mL + (j-1)B) \longrightarrow \mathcal{O}_X(K_X + mL + jB) \longrightarrow \mathcal{O}_B(K_B + mL + (j-1)B) \longrightarrow 0.$$

By applying Kodaira-Kawamata-Viehweg vanishing on B,

$$H^i(X, \mathcal{O}_X(K_X + mL + (j-1)L)) \xrightarrow{\sim} H^i(X, \mathcal{O}_X(K_X + mL + jL)),$$

for $i > 1$, $j > 0$. Hence Serre vanishing implies our claim 1.

Let C be an irreducible curve on X and I the defining ideal of C. We must show $(L \cdot C) \geq 0$.

Claim 2. To show $(L \cdot C) \geq 0$, we may assume C is non-singular.

There is a sequence of blowing-ups

$$X_k \xrightarrow{f_k} X_{k-1} \xrightarrow{f_{k-1}} \cdots \xrightarrow{f_1} X_0 = X$$

such that $X_i \xrightarrow{f_i} X_{i-1}$ is a blowing-up at some point of X_{i-1} and proper transform C_k of C by $f_k \cdots f_1$ is non-singular. Let C_i the proper transform of C by $f_i \cdots f_1$, L_i a pull-back of L by $f_i \cdots f_1$ and E_i the exceptional locus of f_i. Then, since $K_{X_i} + mL_i = f_i^*(K_{X_{i-1}} + mL_{i-1}) + (d-1)E_i$ and $R^j(f_i)_*(\mathcal{O}_{X_i}((d-1)E_i)) = 0$ for $j > 0$. Therefore, we get $H^1(X_i, \mathcal{O}_{X_i}(K_{X_i} + mL_i)) \simeq H^1(X_{i-1}, \mathcal{O}_{X_{i-1}}(K_{X_{i-1}} + mL_{i-1}))$, which implies claim 2.

We set $C_a = \mathrm{Spec}(\mathcal{O}_X/I^{a+1})$. Considering the exact sequence

$$0 \longrightarrow \mathcal{O}_X(K_X + mkL) \otimes I^m \longrightarrow \mathcal{O}_X(K_X + mkL) \longrightarrow \mathcal{O}_{C_{m-1}}(L_X + mkL) \longrightarrow 0$$

and claim 1, we get the inequality

$$\dim H^1(C_{m-1}, \mathcal{O}_{C_{m-1}}(L_X + mkL)) \leq \dim H^1(\mathcal{O}_X(K_X + mkL)) + \dim H^2(\mathcal{O}_X(K_X + mkL) \otimes I^m).$$

Hence,

$$-\frac{\chi(C_{m-1}, \mathcal{O}_{C_{m-1}}(L_X + mkL))}{m^d} \leq \frac{\dim H^1(\mathcal{O}_X(K_X + mkL))}{m^d} + \frac{\dim H^2(\mathcal{O}_X(K_X + mkL) \otimes I^m)}{m^d}.$$

Taking superior limt of the above inequality, we obtain

$$-\lim_{m\to\infty}\frac{\chi(C_{m-1},\mathcal{O}_{C_{m-1}}(L_X+mkL))}{m^d}\leq \limsup_{m\to\infty}\frac{\dim H^2(\mathcal{O}_X(K_X+mkL)\otimes I^m)}{m^d}.$$

Claim 3. $\displaystyle\lim_{m\to\infty}\frac{\chi(C_{m-1},\mathcal{O}_{C_{m-1}}(L_X+mkL))}{m^d}=\frac{k(L\cdot C)}{(d-1)!}+\frac{\deg(I/I^2)}{d!}.$

Considering the exact sequence

$$0\longrightarrow \mathrm{Sym}^a(I/I^2)\otimes\mathcal{O}_C(K_X+mkL)\longrightarrow\mathcal{O}_{C_a}(K_X+mkL)\longrightarrow\mathcal{O}_{C_{a-1}}(K_X+mkL)\longrightarrow 0,$$

we have

$$\begin{aligned}\chi(C_{m-1},\mathcal{O}_{C_{m-1}}(K_X+mkL))&=\sum_{a=0}^{m-1}\chi(\mathrm{Sym}^a(I/I^2)\otimes\mathcal{O}_C(K_X+mkL))\\&=\sum_{a=0}^{m-1}\{\mathrm{rank}(\mathrm{Sym}^a(I/I^2))(K_X+mkL\cdot C)+\deg(\mathrm{Sym}^a(I/I^2))\}\\&=\sum_{a=0}^{m-1}\{\binom{a+d-2}{d-2}(K_X+mkL\cdot C)+\frac{a}{d-1}\binom{a+d-2}{d-2}\deg(I/I^2)\}.\end{aligned}$$

Hence, by easy calculation, we get claim 3.

Claim 4. There exits a constant M such that M is independent of k and

$$\dim H^2(\mathcal{O}_X(K_X+mkL)\otimes I^m)\leq M\cdot m^d.$$

Take a general member B of $|A|$ such that $\mathcal{O}_B\otimes I^m=p_1^m\cdots p_t^m$ for all $m>0$, where $p_1,\cdots p_r$ are maximal ideals of $\mathcal{O}_B$. Set $P=p_1\cdots p_r$. Let $g:B'\longrightarrow B$ be a blowing-up by ideal P and l_i's exceptional set of g. There is a constant T such that $Tg^*(A)-\sum l_i$ is ample on B'. For $i>T(m-d+2)$, since

$$\begin{aligned}g^*(K_B+(i-1)A+mkL)-\sum ml_i=K_{B'}&+(m-d+2)(Tg^*(A)-\sum l_i)\\&+(i-1-T(m-d+2))g^*(A)+mkL\end{aligned}$$

and

$$(m-d+2)(Tg^*(A)-\sum l_i)+(i-1-T(m-d+2))g^*(A)+mkL$$

is ample, we get

$$H^1(B',\mathcal{O}_{B'}(g^*(K_B+(i-1)A+mkL)-\sum ml_i)))=0.$$

Hence the above vanishing and $R^{\cdot}g_*\mathcal{O}(-\sum ml_i)=P^m$ imply

$$H^1(B,\mathcal{O}_B(K_B+(i-1)A+mkL)\otimes P^m)=0\quad\text{for all } i>T(m-d+2).$$

From the exact sequence

$$0\longrightarrow\mathcal{O}_B(K_B+(i-1)A+mkL)\otimes P^m\longrightarrow\mathcal{O}_B(K_B+(i-1)A+mkL)\longrightarrow\mathcal{O}_{m(p_1+\cdots+p_r)}\longrightarrow 0,$$

we obtain the surjective homomorphism $H^0(\mathcal{O}_m(p_1+\cdots+p_r))\longrightarrow H^1(B,\mathcal{O}_B(K_B+(i-1)A+mkL)\otimes P^m)$ and $H^2(B,\mathcal{O}_B(K_B+(i-1)A+mkL)\otimes P^m)=0$ for $i>0$. Therefore, the exact sequence

$$0\longrightarrow\mathcal{O}_X(K_X+(i-1)A+mkL)\otimes I^m\longrightarrow\mathcal{O}_X(K_X+iA+mkL)\longrightarrow\mathcal{O}_B(K_B+(i-1)A+mkl)\otimes P^m\longrightarrow 0,$$

implies an inequality

$$\dim H^2(X,\mathcal{O}_X(K_X+(i-1)A+mkL)\otimes I^m)\leq\sum_{i=1}^{T(m-d+2)}\dim H^0(\mathcal{O}_{m(p_1+\cdots+p_r)}).$$

Thus, we get claim 4.

By claim 3 and 4, we have an inequality $-\dfrac{k(L\cdot C)}{(d-1)!}-\dfrac{\deg(I/I^2)}{d!}\leq M$. Since k is arbitrary positive integer, $(L\cdot C)$ must be non-negative, which proves our Theorem.

On Finite Galois Covering Germs

Makoto Namba

Dedicated to Professor Shingo Murakami on his sixtieth birthday

Introduction. We denote by $\mathbb{C}^n$ the n-th Cartesian product of the complex plane $\mathbb{C}$. Let $W = (W,0)$ be the germ of open balls in $\mathbb{C}^n$ with the center $0 = (0,\cdots,0)$. A finite covering germ is, by definition, a germ $\pi : X \longrightarrow W$ of surjective proper finite holomorphic mappings, where $X = (X,p)$ is a germ of irreducible normal complex spaces.

Every normal singularity (X,p) has the structure of a finite covering germ $\pi : X \longrightarrow W$, (see Gunning-Rossi[4]).

Finite covering germs were discussed in Gunning[3] from the ring theoretic point of view.

In this paper, we introduce the notion of finite Galois covering germs and prove two basic theorems (Theorems 2 and 3 below) on it.

1. Some definitions. Let M be an n-dimensional (connected) complex manifold. A finite covering of M is, by definition, a surjective proper finite holomorphic mapping $\pi : X \longrightarrow M$, where X is an irreducible normal complex space. Let $\pi : X \longrightarrow M$ and $\mu : Y \longrightarrow M$ be finite coverings of M. A morphism (resp. an isomorphism) of π to μ is, by definition, a surjective holomorphic (resp. biholomorphic) mapping φ : $X \longrightarrow Y$ such that $\mu\varphi = \pi$. We denote by G_π the group of all automorphisms

of π and call it the <u>automorphism group of</u> π. G_π acts on each fiber of π.

A finite covering $\pi : X \longrightarrow M$ is called a <u>finite Galois covering</u> if G_π acts transitively on every fiber of π. In this case, the quotient complex space X/G_π (see Cartan[1]) is biholomorphic to M.

For a finite covering $\pi : X \longrightarrow M$, put

$$R_\pi = \left\{ p \in X \;\middle|\; \pi \text{ is not biholomorphic around } p \right\},$$

$$B_\pi = \pi(R_\pi).$$

They are hypersurfaces (i.e. codimension 1 at every point) of X and M, respectively and are called the <u>ramification locus</u> and the <u>branch locus of</u> π, respectively.

Let B be a hypersurface of M. A finite covering $\pi : X \longrightarrow M$ is said to <u>branch at most at</u> B if the branch locus B_π of π is contained in B. In this case, the restriction

$$\pi' : X - \pi^{-1}(B) \longrightarrow M - B$$

of π is an unbranched covering. The mapping degree of π' is called the <u>degree of</u> π and is denoted by $\deg \pi$.

By a property of normal complex spaces, we have easily (see Namba[5])

<u>Proposition 1.</u> (1) $G_\pi \simeq G_{\pi'}$ naturally. (2) π is a Galois covering if and only if π' is a Galois covering.

<u>Corollary.</u> $\#G_\pi \leqslant \deg \pi$, where $\#G_\pi$ is the order of the group G_π. Moreover, the equality holds if and only if π is a Galois covering.

The following theorem is a deep one.

Theorem 1 (Grauert-Remmert[2]). If $\pi' : X' \longrightarrow M - B$ is an unbranched finite covering, then there exists a unique (up to isomorphisms) finite covering $\pi : X \longrightarrow M$ which extends π'.

Take a point $q_o \in M - B$ and fix it. We denote by $\pi_1(M - B, q_o)$ the fundamental group of $M - B$ with the reference point q_o.

Corollary. There is a one-to-one correspondence between isomorphism classes of finite (resp. Galois) coverings $\pi : X \longrightarrow M$ which branches at most at B and the set of all conjugacy classes of subgroups (resp. normal subgroups) H of $\pi_1(M - B, q_o)$ of finite index. If H is normal, then π corresponding to H satisfies

$$G_\pi \simeq \pi_1(M - B, q_o)/H.$$

Example 1. Put $X = \mathbb{C}^n$, $M = \mathbb{C}^n$ and

$$\pi : (x_1, \cdots, x_n) \in \mathbb{C}^n \longmapsto (a_1, \cdots, a_n) \in \mathbb{C}^n,$$

where

$$a_1 = -(x_1 + \cdots + x_n),$$
$$a_2 = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a_n = (-1)^n x_1 \cdots x_n.$$

In other words, x_j $(1 \leqslant j \leqslant n)$ are the roots of the equation

$$x^n + a_1x^{n-1} + \cdots + a_n = 0.$$

Then π is a Galois covering of $M = \mathbb{C}^n$ such that (i) $B_\pi = \Delta$ is the discriminant locus and (ii) $G_\pi \simeq S_n$ (the n-th symmetric group).

We may identify G_π and S_n through the isomorphism. S_n is then regarded as a finite subgroup of the general linear group $GL(n, \mathbb{C})$.

Example 2. We regard S_n as a finite subgroup of $GL(n,\mathbb{C})$ as in Example 1. Put $Y = \mathbb{C}^n$. Let G be a subgroup of S_n. The quotient space Y/G is an irreducible normal complex space and the canonical projection

$$\mu : Y \longrightarrow Y/G = N$$

is a holomorphic mapping. Let

$$\alpha : M \longrightarrow N$$

be a resolution of singularity of N. Then the finite Galois covering $\pi : X \longrightarrow M$ of M, defined by the following diagram, satisfies $G_\pi \simeq G$:

$$\begin{tikzcd} X \arrow[r, "\varsigma"'] \arrow[d, "\pi"'] & M \times_N Y \arrow[r] \arrow[d] & Y \arrow[d, "\mu"] \\ M \arrow[r, "id"'] & M \arrow[r, "\alpha"'] & N. \end{tikzcd}$$

Here, $M \times_N Y$ is the fiber product, ς is the normalization and id is the identity mapping.

2. Finite Galois covering germs. Now, let $W = (W,0)$ be the germ of open balls in $\mathbb{C}^n$ with the center $0 = (0,\cdots,0)$. Let $\pi : X \longrightarrow W$ be a finite covering germ (see Introduction). Every notion in §1 can be easily extended to finite covering germs. In particular, a finite covering germ $\pi : X \longrightarrow W$ is called a finite Galois covering germ if G_π acts transitively on every fiber of π. Also, a similar assertion to Corollary to Theorem 1 holds in the case of finite covering germs, if $\pi_1(M - B, q_o)$ is replaced by the local fundamental group $\pi_{1,loc,0}(W - B)$ of $W - B$ at 0.

Example 3. Let $\pi_o : X \longrightarrow W$ be the restriction of the covering $\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ in Example 1 to $W = (W,0)$ and $X = (X,0) = \pi^{-1}(W)$. Then π_o is a finite Galois covering germ such that $G_{\pi_o} \simeq S_n$.

There exist a lot of finite Galois covering germs in the following sense:

Theorem 2. For $n \geqslant 2$, let $W = (W, 0)$ be the germ of balls in $\mathbb{C}^n$ with the center 0. For every finite group G, there exists a finite Galois covering germ $\pi : X \longrightarrow W$ such that $G_\pi \simeq G$.

Proof. Case 1. We first prove the theorem for the case $n = 2$. Let W be a ball in $\mathbb{C}^2$ with the center 0. Let L_j $(1 \leqslant j \leqslant s)$ be mutually distinct (complex) lines in $\mathbb{C}^2$ passing through 0. Put $D_j = L_j \cap W$ $(1 \leqslant j \leqslant s)$ and

$$B = D_1 \cap \cdots \cap D_s,$$

(see Figure 1). Figure 1

Take a point $q_o \in M - B$ and fix it. Let γ_j be a loop in $M - B$ starting from q_o and rounding $D_j - 0$ once counterclockwisely as in Figure 2. We identify γ_j with its homotopy class. Figure 2

Then, as is well known, $\pi_1(W - B,\ q_o)$ is a group generated by $\gamma_1, \cdots, \gamma_s$ with the generating relations

$$\gamma_j \delta = \delta \gamma_j \quad (1 \leqslant j \leqslant s),$$

where $\delta = \gamma_1 \cdots \gamma_s$.

Let F_{s-1} be the free group of $(s-1)$-letters $b_1, \cdots, b_{s-1}$. Put $b_s = (b_1 \cdots b_{s-1})^{-1}$. Then there is the surjective homomorphism

$$\Phi : \pi_1(W - B,\ q_o) \longrightarrow F_{s-1}$$

defined by $\Phi(\gamma_j) = b_j$ $(1 \leqslant j \leqslant s)$.

For any finite group G, there is a surjective homomorphism

$$\Psi : F_{s-1} \longrightarrow G$$

for a sufficiently large s.

Now, the kernel K of the surjective homomorphism

$$\Psi\Phi : \pi_1(W - B, q_o) \longrightarrow G$$

has a finite index such that

$$\pi_1(W - B, q_o)/K \simeq G.$$

The finite Galois covering $\pi : X \longrightarrow W$ corresponding to K in Corollary to Theorem 1 satisfies $G_\pi \simeq G$.

The finite Galois covering germ determined by π is a desired one.

Case 2. Next, we prove the theorem for the case $n \geqslant 3$. Let W be a ball in $\mathbb{C}^n$ with the center 0. Let P and Q be a 2-plane and an $(n - 2)$-plane in $\mathbb{C}^n$, respectively, passing through 0 such that $P \cap Q = \{0\}$. Let H_j $(1 \leqslant j \leqslant s)$ be mutually distinct hyperplanes in $\mathbb{C}^n$ passing through 0 and containing Q (see Figure 3). Figure 3

Put $D_j = H_j \cap W$ $(1 \leqslant j \leqslant s)$ and

$$B = D_1 \cap \cdots \cap D_s.$$

Then $W - B$ and $W \cap P - B \cap P$ are homotopic. Hence, by Case 1, taking sufficiently large s, there exists a normal subgroup K of $\pi_1(W - B, q_o)$ of finite index such that

$$\pi_1(W - B, q_o)/K \simeq G.$$

The rest of the proof is similar to Case 1. q.e.d.

Now, we give a method of concrete constructions of every finite Galois covering germ. Our method is suggested by Professor Enoki and is different from and simpler than Namba[6] in which finite Galois coverings of projective manifolds were treated.

Theorem 3. Let $\pi : X \longrightarrow W$ be a finite Galois covering germ. Put $m = \deg \pi$. Then there exists a germ $f : W \longrightarrow \mathbb{C}^m$ of holomorphic mappings and a finite subgroup G of S_m with $G \simeq G_\pi$ such that π is obtained by the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow[\rho]{} & W\times_N Y & \longrightarrow & \mathbb{C}^m = Y \\ \pi\downarrow & & \downarrow & & \downarrow\mu \\ W & \xrightarrow[id]{} & W & \xrightarrow[f]{} & \mathbb{C}^m/G = N, \end{array}$$

where $W\times_N Y$ is the fiber product, ρ is the normalization and id is the identity mapping. Here S_m is regarded as a finite subgroup of $GL(m,\mathbb{C})$ as in Example 1.

Proof. We may assume that W is a small ball in $\mathbb{C}^n$ with the center 0. Take a point $q_o \in W - B$ and put

$$\pi^{-1}(q_o) = \{p_1, \cdots, p_m\}.$$

Put
$$G_\pi = \{\sigma_1 = 1, \sigma_2, \cdots, \sigma_m\}.$$

Note that X is a Stein space. Let h be a holomorphic function on X such that

$$h(p_j) \neq h(p_k) \quad \text{for} \quad j \neq k \qquad \cdots(1)$$

Put

$$h_j = \sigma_j^* h = h\cdot\sigma_j \quad (1 \leqq j \leqq m).$$

Let $F : X \longrightarrow \mathbb{C}^m$ be the holomorphic mapping defined by

$$F(p) = (h_1(p), \cdots, h_m(p)).$$

Then, for $\sigma \in G$,

$$\begin{aligned}(\sigma^*F)(p) = F(\sigma(p)) &= (h_1(\sigma(p)), \cdots, h_m(\sigma(p)) \\ &= (h(\sigma(p)), h(\sigma_2\sigma(p)), \cdots, h(\sigma_m\sigma(p)) \\ &= (h_{k(1)}(p), h_{k(2)}(p), \cdots, h_{k(m)}(p)) \qquad \cdots(2)\end{aligned}$$

Thus σ gives the permutation

$$R(\sigma) = \begin{pmatrix} 1 & 2 & \cdots & m \\ k(1) & k(2) & \cdots & k(m) \end{pmatrix}.$$

The correspondence

$$R : \sigma \longmapsto R(\sigma)$$

is then an isomorphism of G_{π} onto a subgroup G of S_m. (2) can be rewritten as

$$\sigma^{*}F = R(\sigma)F \quad \text{for all} \quad \sigma \in G . \tag{3}$$

Hence F induces a holomorphic mapping $f : W \longrightarrow \mathbb{C}^m/G = N$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\pi"'] & \mathbb{C}^m = Y \arrow[d, "\mu"] \\ W \arrow[r, "f"'] & \mathbb{C}^m/G = N. \end{tikzcd}$$

By the assumption (1), we can easily show that f has the following two properties:

(i) $f(W) \not\subset \text{Fix } G$, where Fix G is the union of the fixed points of all elements of G except the identity and

(ii) f is not decomposed as follows:

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "\mathrm{id}"'] & \mathbb{C}^m/H \arrow[d, "\nu"] \\ W \arrow[r, "f"'] & \mathbb{C}^m/G, \end{tikzcd}$$

where H ($\neq G$) is a subgroup of G, ν is the canonical projection and f' is a holomorphic mapping.

A holomorphic mapping f with the properties (i) and (ii) is said to be <u>G-indecomposable</u> (see Namba[6]). For such a mapping f, the fiber product $W \times_N Y$ is irreducible and the finite Galois covering $\pi_o : X_o \longrightarrow W$ defined by the commutative diagram

$$\begin{tikzcd} X_o \arrow[r, "\varrho"'] \arrow[d, "\pi_o"'] & W \times_N Y \arrow[r] \arrow[d] & \mathbb{C}^m = Y \arrow[d, "\mu"] \\ W \arrow[r, "\mathrm{id}"'] & W \arrow[r, "f"'] & \mathbb{C}^m/G = N \end{tikzcd}$$

satisfies $G_{\pi_o} \simeq G$. Now, we can easily show that π is isomorphic to π_o, (see Namba[6]). q.e.d.

Remark. (1) $f(0)$ is not necessarily equal to $\mu(0)$, where 0 is the origin of $\mathbb{C}^m$. (2) A similar theorem to Theorem 3 holds for finite Galois coverings of a Stein manifold.

Problem. Characterize normal singularities (X,p) which has the structure of a finite Galois covering germs $\pi : X \longrightarrow W$.

References

[1] H. Cartan: Quotient d'un espace analytique par un groupe d'automorphisms, Algebraic Geometry and Topology, A sympos. in honor of S. Lefschetz, Princeton Univ. Press, (1957), 90-102.

[2] H. Grauert and R. Remmert: Komplexe Raume, Math. Ann. 136 (1958), 245-318.

[3] R. Gunning: Lectures on complex analytic varieties: Finite analytic mappings, Math. Notes 14 (1974), Princeton Univ. Press.

[4] R. Gunning and H. Rossi: Analytic functions of several complex variables, Prentice-Hall, New York, 1965.

[5] M. Namba: Branched coverings and algebraic functions, Pitman Research Notes in Math. 161, 1987.

[6] M. Namba: On finite Galois coverings of projective manifolds, J. Math. Soc. Japan, 41 (1989), 391-403.

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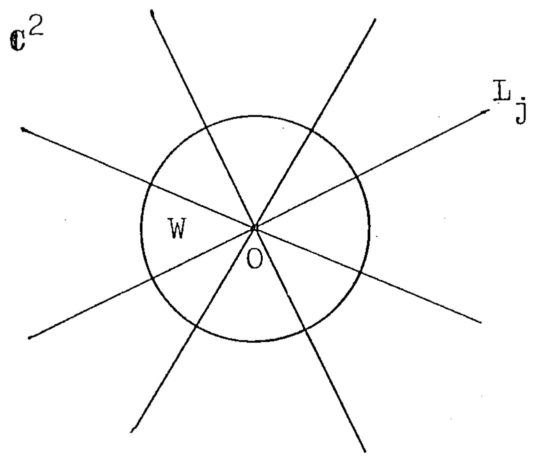

Figure 1

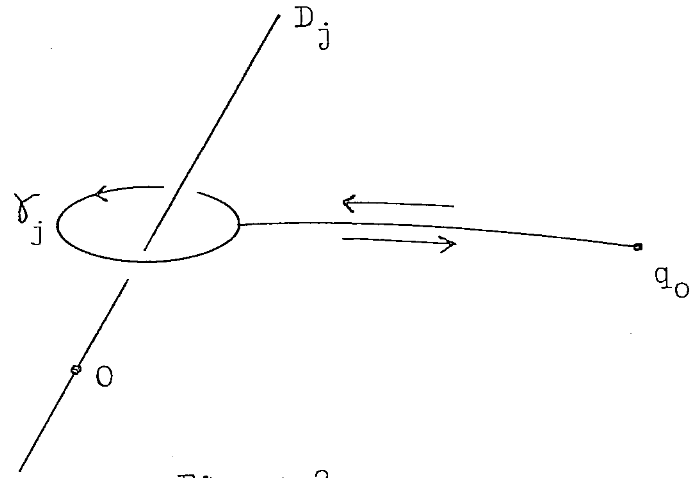

Figure 2

Finite Galois covering germs

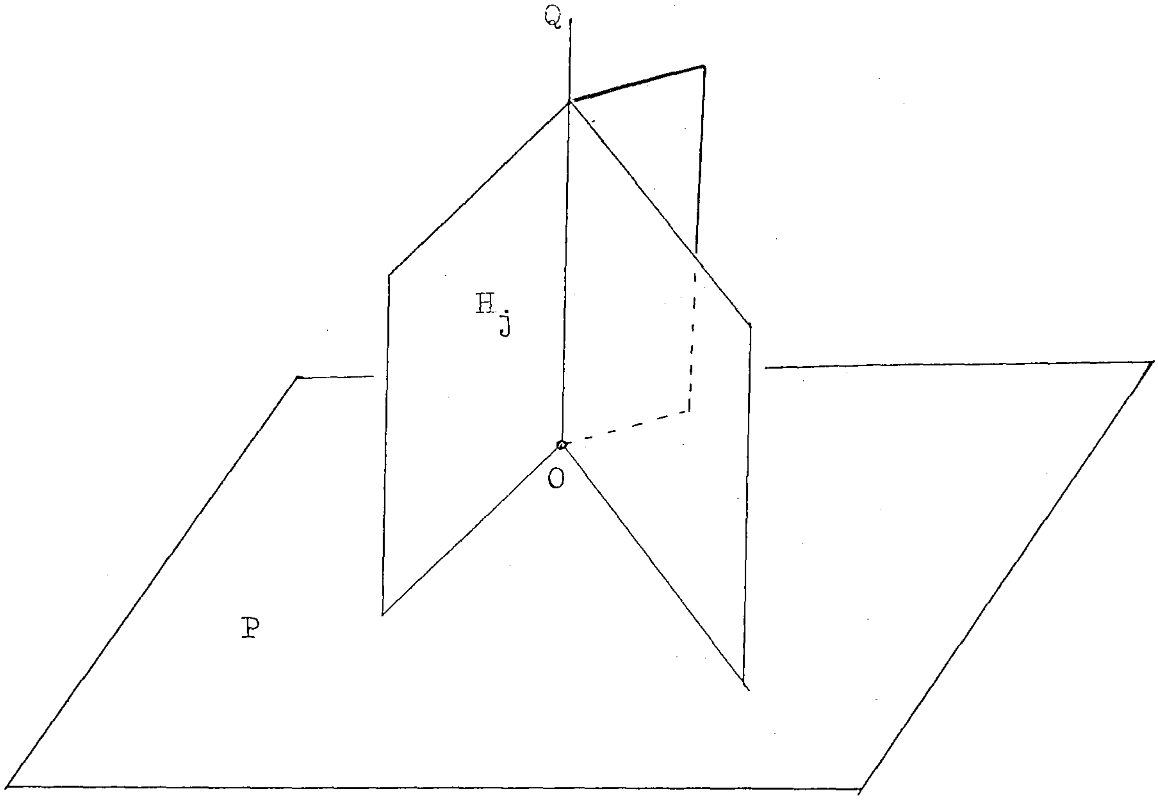

Figure 3

Finite Galois Covering Gemi

Zariski-decomposition Problem for Pseudo-effective Divisors

Noboru NAKAYAMA

Introduction. In this paper, we shall study some numerical properties of pseudo-effective divisors on projective complex manifolds. Here the "numerical properties" are the properties which depend only on the first Chern classes of the divisors. In the paper of [Z], Zariski found the so-called the "Zariski-decomposition" $D = P + N$ for an effective divisor D on a projective smooth surface, where P is a nef $\mathbb{Q}$-divisor, N is an effective $\mathbb{Q}$-divisor with negative intersection matrix, and $P \cdot N = 0$. The construction of the decomposition depends only on the intersection numbers of D with irreducible curves on the surface. Fujita [F1] showed the similar decomposition exist for all pseudo-effective divisors on surfaces. The Zariski-decomposition is very useful to study the projective surfaces and open surfaces. In [F2], Fujita generalize the notion of Zariski-decomposition to higher dimensional case and conjectured the existence of the decomposition for pseudo-effective divisors. In higher dimensional case, one cannot obtain a desired decomposition (namely the positive part is nef) on the fixed manifold, in general. So we must blow-up the manifold. His conjecture is the existence of such a nice blowing-up. On the other hand, Cutkosky [C] found an example where the negative part of the Zariski-decomposition is not a $\mathbb{Q}$-divisor. Matsuda[Ma] also studied the Zariski-decomposition in Fujita's sense and considered the σ-decomposition (see §1). But in his paper, the negative part of the

σ-decomposition may have infinitely many components. If the Zariski-decomposition of the canonical divisor of a projective manifold of general type exists, then the canonical ring is finitely generated over $\mathbb{C}$, by [K1]. Therefore, for example, the Flip Conjecture (the existence of the flip) in the minimal model theory (cf. [KMM]) follows from the existence of the relative Zariski-decomposition of the canonical divisor.

In §1, we shall define the σ-decomposition $D = P_\sigma(D) + N_\sigma(D)$ for pseudo-effective divisors D on a projective complex manifold X. Here $N_\sigma(D)$ is an effective $\mathbb{R}$-divisor determined by the first Chern class of D. We shall show the following properties ((1.9)):

(a) $c_1(P_\sigma(D)) \in \overline{\mathrm{Mv}}(X)$, and

(b) if $c_1(D - \Delta) \in \overline{\mathrm{Mv}}(X)$ for some effective $\mathbb{R}$-divisor Δ, then $\Delta \geq N_\sigma(D)$.

Here $\overline{\mathrm{Mv}}(X)$ is the movable cone of X which is the closure of the cone in $H^2(X, \mathbb{R})$ generated by the first Chern classes of fixed part free effective divisors on X (see (1.8)). If X is a surface, then the σ-decomposition is nothing but the Zariski-decomposition, since the movable cone $\overline{\mathrm{Mv}}(X)$ is the nef cone in this case. We now formulate the Zariski-decomposition Problem as follows:

Problem. For a pseudo-effective $\mathbb{R}$-divisor D on X, does there exist a modification $\mu : Y \longrightarrow X$ such that $P_\sigma(\mu^*D)$ is nef ?

If such a modification exists, then the decomposition $\mu^*D = P_\sigma(\mu^*D) + N_\sigma(\mu^*D)$ is said to be the Zariski-decomposition of D . This is actually in the sense of Fujita [F2].

In §2, we define the ν-decomposition $D = P_\nu(D) + N_\nu(D)$ which satisfies the following two properties:

(c) $c_1(P_\nu(D)) \in NMv(X)$, and

(d) if $c_1(D - \Delta) \in NMv(X)$ for some effective $\mathbb{R}$-divisor Δ , then $\Delta \geq N_\nu(D)$.

Here $NMv(X)$ is the cone consisting of the first Chern classes of pseudo-effective divisors whose restriction to any prime divisors on X are still pseudo-effective (cf. §2). The σ-decomposition and the ν-decomposition of the given divisor are different in general, but the calculation of the ν-decomposition is easier than that of the σ-decomposition. We obtain the Zariski-decomposition for suitable divisors using the ν-decomposition in §4. §3 is devoted to the study of the relative version of σ-decomposition.

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§0. Preliminaries.

We shall use the similar notation as in [KMM] and [N1]. Let X

be an n-dimensional projective complex manifold. We denote by $N^1(X)$ the real vector subspace in $H^2(X, \mathbb{R})$ generated by the first Chern classes $c_1(L)_{\mathbb{R}}$ of line bundles L on X. Note that $c_1(L)_{\mathbb{R}} = 0$ if and only if $L \cdot C = 0$ for any irreducible curves C. We denote the $c_1(L)_{\mathbb{R}}$ by $c_1(L)$, simply. Let $\mathrm{Div}(X)$ be the divisor group of X. An <u>$\mathbb{R}$-divisor</u> D should be an element of $\mathrm{Div}(X) \otimes \mathbb{R}$. Let $D = \sum a_j \Gamma_j$ be the irreducible decomposition of an $\mathbb{R}$-divisor D. D is called <u>effective</u>, if $a_j \geq 0$ for all j. Let $\mathrm{Eff}(X)$ be the cone in $N^1(X)$ generated by the first Chern classes of effective $\mathbb{R}$-divisors and let us denote its closure by $PE(X)$, which is called the <u>pseudo-effective cone</u>. The interior of the cone $PE(X)$ is denoted by $\mathrm{Big}(X)$, which is called the <u>big cone</u>. An $\mathbb{R}$-divisor D is called pseudo-effective, (resp. <u>big</u>), if $c_1(D) \in PE(X)$ (resp. $c_1(D) \in \mathrm{Big}(X)$). Then we have:

Lemma (0.1). *The following two conditions are equivalent:*

(1) B *is a big* $\mathbb{R}$*-divisor.*

(2) $\limsup_{m \to \infty} h^0(X, [mB])/m^n > 0$,

where $[mB]$ *is the integral part of* mB .

Let $\mathrm{Amp}(X)$ be the cone in $N^1(X)$ generated by the first Chern classes of ample line bundles on X. Then $\mathrm{Amp}(X)$ is an open convex cone in $N^1(X)$ which is called the <u>ample cone</u> of X. The <u>nef cone</u> $\mathrm{Nef}(X)$ should be the closure of the $\mathrm{Amp}(X)$. An

$\mathbb{R}$-divisor D is called ample (resp. nef), if $c_1(D) \in \mathrm{Amp}(X)$ (resp. $c_1(D) \in \mathrm{Nef}(X)$). Then we have:

Lemma (0.2). *Let* A *be an ample* $\mathbb{R}$*-divisor on* X *. Then* A *is numerically equivalent to* $\sum s_j H_j$ *, for some ample* $\mathbb{Q}$*-divisors* H_j *and positive real numbers* s_j *.*

Proof. Let $\Gamma_1, \Gamma_2, \cdots \Gamma_k$ be the irreducible components of A whose coefficients are irrational numbers. Since the ampleness is an open condition, $[mA]$ and $[mA] + k\Gamma_j$ $(1 \le j \le k)$ are ample for large integer m . We may assume that $\langle mA \rangle := mA - [mA] = \sum_{j=1}^{k} t_j \Gamma_j$, for $0 < t_j < 1$. Thus $mA = \sum_{j=1}^{k} (t_j/k)([mA] + k\Gamma_j) + (1 - \sum_{j=1}^{k} (t_j/k))[mA]$.

Q.E.D.

§1. σ-decomposition.

Let Γ be a prime divisor of X and let B be a big $\mathbb{R}$-divisor on X . We define $\sigma_\Gamma(B)$ (resp. $\sigma_\Gamma(B)_{\mathbb{Q}}$) to be the following number:

$$\inf\{\mathrm{mult}_\Gamma(\Delta) \mid \Delta \text{ is an effective } \mathbb{R}\text{-divisor with } c_1(\Delta) = c_1(B) \text{ (resp. } \Delta \sim_{\mathbb{Q}} B)\},$$

where the symbol $\sim_{\mathbb{Q}}$ means the $\mathbb{Q}$-linear equivalence relation.

Lemma (1.1). *(1)* $\lim_{\varepsilon \downarrow 0} \sigma_\Gamma(B + \varepsilon A)_{\mathbb{Q}} = \sigma_\Gamma(B)_{\mathbb{Q}}$, *for any ample* $\mathbb{R}$*-divisor* A .

(2) $\sigma_\Gamma(B)_{\mathbb{Q}} = \sigma_\Gamma(B)$.

(3) $\sigma_\Gamma(B_1 + B_2) \le \sigma_\Gamma(B_1) + \sigma_\Gamma(B_2)$, *for any big* $\mathbb{R}$*-divisors* B_1 *and* B_2 .

Proof. (1). It is easy to see that if $0 \le \varepsilon_1 < \varepsilon_2$, then $\sigma_\Gamma(B + \varepsilon_1 A)_{\mathbb{Q}} \ge \sigma_\Gamma(B + \varepsilon_2 A)_{\mathbb{Q}}$. Thus $\lim_{\varepsilon \downarrow 0} \sigma_\Gamma(B + \varepsilon A)_{\mathbb{Q}} \le \sigma_\Gamma(B)_{\mathbb{Q}}$. On the other hand, by **(0.1)**, there are an effective $\mathbb{R}$-divisor Δ and a positive number δ such that $B \sim_{\mathbb{Q}} \delta A + \Delta$. Therefore we have $(1 + \varepsilon)B \sim_{\mathbb{Q}} B + \varepsilon\delta A + \varepsilon\Delta$. Thus the inequality $(1 + \varepsilon)\sigma_\Gamma(B)_{\mathbb{Q}} \le \sigma_\Gamma(B + \varepsilon\delta A)_{\mathbb{Q}} + \varepsilon \mathrm{mult}_\Gamma(\Delta)$ holds. Taking $\varepsilon \downarrow 0$, we have $\sigma_\Gamma(B)_{\mathbb{Q}} \le \lim_{\varepsilon \downarrow 0} \sigma_\Gamma(B + \varepsilon A)_{\mathbb{Q}}$.

(2). By definition, we have only to prove $\sigma_\Gamma(B)_{\mathbb{Q}} \le \sigma_\Gamma(B)$. Let Δ be an effective $\mathbb{R}$-divisor such that $c_1(\Delta) = c_1(B)$. Then $B + A - \Delta$ is ample for any ample $\mathbb{R}$-divisor A . Thus $\sigma_\Gamma(B + A)_{\mathbb{Q}} \le \mathrm{mult}_\Gamma(\Delta)$. Taking A very small, we have $\sigma_\Gamma(B)_{\mathbb{Q}} \le \mathrm{mult}_\Gamma(\Delta)$ by (1). Therefore $\sigma_\Gamma(B)_{\mathbb{Q}} \le \sigma_\Gamma(B)$.

(3) is obvious. Q.E.D.

Let D be a pseudo-effective $\mathbb{R}$-divisor on X .

Lemma (1.2). *(1)* $\lim_{\varepsilon \downarrow 0} \sigma_\Gamma(D + \varepsilon A) < +\infty$ *for any ample* $\mathbb{R}$*-divisor* A .

(2) The number $\lim_{\varepsilon \downarrow 0} \sigma_\Gamma(D + \varepsilon A)$ *does not depend on the choice of* A .

Proof. (1). We have easily $(D + \varepsilon A - \sigma_\Gamma(D + \varepsilon A)\Gamma)\cdot A^{n-1} \geq 0$ for any $\varepsilon > 0$. Hence $\sigma_\Gamma(D + \varepsilon A) \leq (D + \varepsilon A)\cdot A^{n-1}/\Gamma\cdot A^{n-1}$. Therefore $\lim_{\varepsilon\downarrow 0} \sigma_\Gamma(D + \varepsilon A)$ is bounded.

(2). Let A' be another ample $\mathbb{R}$-divisor. Then there are an effective $\mathbb{R}$-divisor Δ and a positive number δ such that $c_1(A') = c_1(\delta A + \Delta)$. Hence $\sigma_\Gamma(D + \varepsilon\delta A) + \varepsilon \mathrm{mult}_\Gamma(\Delta) \geq \sigma_\Gamma(D + \varepsilon A')$. Therefore $\lim_{\varepsilon\downarrow 0} \sigma_\Gamma(D + \varepsilon A) \geq \lim_{\varepsilon\downarrow 0} \sigma_\Gamma(D + \varepsilon A')$. Q.E.D.

In what follows, we denote the $\lim_{\varepsilon\downarrow 0} \sigma_\Gamma(D + \varepsilon A)$ above by $\sigma_\Gamma(D)$. Note that $\sigma_\Gamma(D) = \sigma_\Gamma(D')$, if $c_1(D) = c_1(D')$.

Lemma (1.3). *Let* $\Gamma_1, \Gamma_2, \ldots, \Gamma_\ell$ *be mutually distinct prime divisors and let* $s_1, s_2, \ldots, s_\ell$ *be real numbers with* $0 \leq s_i \leq \sigma_{\Gamma_i}(D)$ *for all* i. *Then* $\sigma_{\Gamma_i}(D - \sum_{j=1}^{\ell} s_j\Gamma_j) = \sigma_{\Gamma_i}(D) - s_i$ *for all* i.

Proof. Take an ample $\mathbb{R}$-divisor A such that $A - \sum \Gamma_j$ is also ample. By definition, for any positive number ε, there exists a positive number δ such that $0 \leq \sigma_{\Gamma_i}(D) - \sigma_{\Gamma_i}(D + \delta A) < \varepsilon$, for all i. Let $E := D - \sum_{j=1}^{\ell} s_j\Gamma_j$. Then we have

$$\begin{aligned} E + (\varepsilon + \delta)A &= D + (\varepsilon + \delta)A - \sum s_j\Gamma_j \\ &= (D + \delta A - \sum \sigma_{\Gamma_j}(D + \delta A)\Gamma_j) + \varepsilon(A - \sum \Gamma_j) \end{aligned}$$

$$+ \left(\sum(\varepsilon - (\sigma_{\Gamma_j}(D) - \sigma_{\Gamma_j}(D + \delta A)))\Gamma_j\right)$$
$$+ \left(\sum(\sigma_{\Gamma_j}(D) - s_j)\Gamma_j\right) .$$

Therefore we have $\varepsilon + \sigma_{\Gamma_j}(D + \delta A) - s_j \geq \sigma_{\Gamma_j}(E + (\varepsilon + \delta)A)$. Taking $\delta \leq \varepsilon$, we get $\varepsilon + \sigma_{\Gamma_j}(D) - s_j \geq \sigma_{\Gamma_j}(E + 2\varepsilon A)$. Hence $\sigma_{\Gamma_j}(D) - s_j \geq \sigma_{\Gamma_j}(E)$. On the other hand, we have $\sigma_{\Gamma_j}(E + \varepsilon A) + s_j \geq \sigma_{\Gamma_j}(D + \varepsilon A)$, since $E + \varepsilon A + \sum s_j\Gamma_j = D + \varepsilon A$. Therefore $\sigma_{\Gamma_j}(D) - s_j = \sigma_{\Gamma_j}(E)$. Q.E.D.

Proposition (1.4). *Let* D *be a pseudo-effective* $\mathbb{R}$*-divisor and let* $\Gamma_1, \Gamma_2, \ldots, \Gamma_\ell$ *be mutually distinct prime divisors on* X *such that* $\sigma_{\Gamma_i}(D) > 0$ *for all* i . *Then* $c_1(\Gamma_1), c_1(\Gamma_2), \ldots, c_1(\Gamma_\ell)$ *are linearly independent in* $N^1(X)$.

Proof. Assume the contrary. Then we may assume that $c_1(\sum_{i=1}^{s} a_i\Gamma_i) = c_1(\sum_{j=s+1}^{\ell} b_j\Gamma_j)$ for some positive numbers a_i and b_j and for some $1 \leq s < \ell$. Take a positive number ε such that $\sigma_{\Gamma_i}(D) > \varepsilon a_i$ for $1 \leq i \leq s$ and that $\sigma_{\Gamma_j}(D) > \varepsilon b_j$ for $s+1 \leq j \leq \ell$. Applying (1.3) to $D - \varepsilon(\sum_{i=1}^{s} a_i\Gamma_i)$ and $D - \varepsilon(\sum_{j=s+1}^{\ell} b_j\Gamma_j)$, we have a contradiction.

Q.E.D.

Corollary (1.5). *For any pseudo-effective* $\mathbb{R}$*-divisor* D , *there exist* *at most finite numbers of prime divisors* Γ *on* X *such that* $\sigma_\Gamma(D) > 0$.

Definition (1.6). For a pseudo-effective $\mathbb{R}$-divisor D , let $N_\sigma(D)$ be the effective $\mathbb{R}$-divisor $\sum \sigma_\Gamma(D)\Gamma$ and set $P_\sigma(D) := D - N_\sigma(D)$. The decomposition $D = P_\sigma(D) + N_\sigma(D)$ is said to be the <u>σ-decomposition</u> of D and $P_\sigma(D)$ and $N_\sigma(D)$ are called the <u>positive</u> and <u>negative</u> <u>parts</u> of the σ-decomposition of D , respectively.

Remark. (1) We have $N_\sigma(B) = \lim_{m\to\infty} (1/m)\cdot|[mB]|_{fix}$ for big divisors B .

(2) If $P_\sigma(D)$ is nef, then the decomposition $D = P_\sigma(D) + N_\sigma(D)$ is the Zariski decomposition in Fujita's sense ([F2]). Therefore we are interested in the following:

Zariski-decomposition Problem. For a pseudo-effective $\mathbb{R}$-divisor D on X , does there exist a modification $\mu : Y \longrightarrow X$ such that $P_\sigma(\mu^* D)$ is nef?

Proposition (1.7). *(1)* $\sigma_\Gamma : PE(X) \longrightarrow \mathbb{R}_{\geq 0}$ *is a lower semi-continuous function. (Therefore it is continuous on* Big(X)).

(2) $\lim_{\varepsilon\downarrow 0} \sigma_\Gamma(D + \varepsilon E) = \sigma_\Gamma(D)$ *for any pseudo-effective* $\mathbb{R}$-*divisors* E *on* X .

(3) *If* $\sigma_\Gamma(D) = 0$, *then for any ample* $\mathbb{R}$-*divisor* A , *there exists an effective* $\mathbb{R}$-*divisor* Δ *with* $c_1(\Delta) = c_1(D + A)$ *and*

$\mathrm{mult}_\Gamma(\Delta) = 0$.

Proof. (1). Let $\{D_n\}$ $(n \in \mathbb{N})$ be a sequence in PE(X) which convergent to D . Assume that $\sigma_\Gamma(D_n) \leq \lambda$ for any n . Take an ample $\mathbb{R}$-divisor A on X . Then for any $\varepsilon > 0$, there is a number n_0 such that $D - D_n + \varepsilon A$ is ample for $n \geq n_0$. Since $D + \varepsilon A = (D - D_n + \varepsilon A) + D_n$, we have $\sigma_\Gamma(D + \varepsilon A) \leq \sigma_\Gamma(D_n) \leq \lambda$.

(2). By (1), we have $\liminf_{\varepsilon \downarrow 0} \sigma_\Gamma(D + \varepsilon E) \geq \sigma_\Gamma(D)$. On the other hand, $\sigma_\Gamma(D + \varepsilon E) \leq \sigma_\Gamma(D) + \varepsilon\sigma_\Gamma(E)$. Therefore we are done.

(3). Let m be a positive number such that $mA + \Gamma$ is ample. For any small $\varepsilon > 0$, there exist positive numbers λ and δ and an effective $\mathbb{R}$-divisor B such that $c_1(B + \delta\Gamma) = c_1(D + \lambda A)$, $\mathrm{mult}_\Gamma(B) = 0$ and that $m\delta + \lambda < \varepsilon$. Then $c_1(B + \delta(mA + \Gamma)) = c_1(D + (m\delta + \lambda)A)$. Thus we can find a desired effective divisor. Q.E.D.

The following definition is found in Kawamata[K2].

Definition (1.8). Let $\acute{\mathrm{Mv}}(X)$ be the cone in $N^1(X)$ generated by the first Chern classes of fixed part free line bundles L (i.e., $|L|_{fix} = 0$). We denote its closure by $\overline{\mathrm{Mv}}(X)$ and denote by $\mathrm{Mv}(X)$ the interior of $\overline{\mathrm{Mv}}(X)$. The cones $\overline{\mathrm{Mv}}(X)$ and $\mathrm{Mv}(X)$ are called the <u>movable cone</u> and the <u>strictly movable cone</u>, respectively.

Proposition (1.9). *(1) For a pseudo-effective $\mathbb{R}$-divisor* D , $N_\sigma(D) = 0$ *if and only if* $c_1(D) \in \overline{\mathrm{Mv}}(X)$.

(2) For a pseudo-effective $\mathbb{R}$-divisor D, if there is an effective $\mathbb{R}$-divisor Δ such that $c_1(D - \Delta) \in \overline{\mathrm{Mv}}(X)$, then $\Delta \geq N_\sigma(D)$.

Proof. (1). Assume that $N_\sigma(D) = 0$. Then by the proof of **(1.7, (3))**, we see that $c_1(D + A) \in \mathrm{M\acute{v}}(X)$, for any ample $\mathbb{R}$-divisor A. Therefore $c_1(D) \in \overline{\mathrm{Mv}}(X)$. The converse is derived from **(1.7, (1))**.

(2). By (1), $N_\sigma(D - \Delta) = 0$. Thus for any prime divisor Γ, $\sigma_\Gamma(D) \leq \sigma_\Gamma(D - \Delta) + \sigma_\Gamma(\Delta) \leq \mathrm{mult}_\Gamma(\Delta)$. Therefore $N_\sigma(D) \leq \Delta$. Q.E.D.

Remark. If X is a surface, then the movable cone is nothing but the nef cone. Therefore by **(1.9)**, we see that the σ-decomposition is nothing but the usual Zariski decomposition (cf. [Z] and [F1]).

Definition (1.10). Let $W \subset X$ be a subvariety with $\mathrm{codim}\, W \geq 2$. For a pseudo-effective $\mathbb{R}$-divisor D, we define $\sigma_W(D)$ to be the following number:

$$\lim_{\varepsilon \downarrow 0} \inf\{\mathrm{mult}_W(\Delta) \mid \Delta \text{ is an effective } \mathbb{R}\text{-divisor with } c_1(\Delta) = c_1(D + \varepsilon A)\},$$

where A is an ample divisor.

From the following **(1.11)**, we see that the number $\sigma_W(D)$ does not depend on the choice of A.

Lemma (1.11). *Let* $Q_W(X) \longrightarrow X$ *be the blowing-up along* W *and let* $Y \longrightarrow Q_W(X)$ *be a resolution of singularities. Then for the main exceptional divisor* E_W *on* Y *, we have* $\sigma_{E_W}(f^*D) = \sigma_W(D)$ *, where* $f : Y \longrightarrow X$ *is the composition.*

Proof. Let Δ be an effective $\mathbb{R}$-divisor on X . Then $\mathrm{mult}_W(\Delta) = \mathrm{mult}_{E_W}(f^*\Delta)$. Thus we have that

$\sigma_W(D) = \lim_{\varepsilon \downarrow 0} \sigma_{E_W}(f^*(D + \varepsilon A)) = \sigma_{E_W}(f^*D)$ by **(1.7, (2))**. Q.E.D.

Lemma (1.12). *(1)* $\sigma_W(D) \le \sigma_x(D)$ *for all* $x \in W$.

(2) There is a countable union S *of proper closed analytic subsets of* W *such that* $\sigma_W(D) = \sigma_x(D)$ *for* $x \in W \setminus S$.

(3) The function $X \ni x \longmapsto \sigma_x(B)$ *is upper semi-continuous for big* $\mathbb{R}$*-divisors* B .

Proof. (1) and (2). Let Δ be an effective $\mathbb{R}$-divisor and $\Delta = \sum r_j \Gamma_j$ be the irreducible decomposition of Δ . Then by definition, $\mathrm{mult}_W(\Delta) = \sum r_j \mathrm{mult}_W(\Gamma_j)$. Therefore $\mathrm{mult}_x(\Delta) \ge \mathrm{mult}_W(\Delta)$ and further there exists a Zariski-open subset U of W such that $\mathrm{mult}_x(\Delta) = \mathrm{mult}_W(\Delta)$ for $x \in U$. Thus we are done.

(3). We have $\sigma_x(B) = \lim_{\varepsilon \downarrow 0} \inf\{\mathrm{mult}_x(\Delta) \mid \Delta \ge 0$ and $c_1(\Delta) = c_1(B)\}$, since B is big. Therefore the upper semi-continuity of $\sigma_x(B)$ is derived from the upper semi-continuity of the function $x \longmapsto \mathrm{mult}_x(\Delta)$.

Q.E.D.

Lemma (1.13). *Let* D *be a pseudo-effective* $\mathbb{R}$*-divisor on* X.

(1) If $f : Y \longrightarrow X$ *be a birational morphism from a projective manifold* Y, *then* $N_\sigma(f^*D) \geq f^*N_\sigma(D)$.

(2) $\sigma_x(D) = \sigma_x(P_\sigma(D)) + \mathrm{mult}_x(N_\sigma(D))$ *for* $x \in X$.

(3) Let $\mu : Q_x(X) \longrightarrow X$ *be the blowing-up at a point* $x \in X$. *Then for any* $y \in f^{-1}(x)$, *we have* $\sigma_y(P_\sigma(\mu^*D)) \leq \sigma_x(P_\sigma(D))$.

Proof. (1). Take an ample divisor A on X. If Δ is an effective $\mathbb{R}$-divisor on Y such that $c_1(\Delta) = c_1(f^*(D + \varepsilon A))$ for some positive ε, then $\Delta = f^*(f_*\Delta)$ and $c_1(f_*\Delta) = c_1(D + \varepsilon A)$. Therefore we have $N_\sigma(f^*(D + \varepsilon A)) \geq f^*N_\sigma(D + \varepsilon A)$. By **(1.7,** (2)**)**, we are done.

(2). Let $E = \mu^{-1}(x)$ be the exceptional divisor for μ (defined in (3)). Then by (1) and **(1.3)**, we have $\sigma_E(\mu^*D) = \sigma_E(\mu^*P_\sigma(D)) + \mathrm{mult}_E(\mu^*N_\sigma(D))$. This is just the desired formula by **(1.11)**.

(3). By (1) and **(1.7)**, it is enough to prove (3) in the case that $c_1(D) \in \mathrm{Mv}(X)$. In this case, (3) is derived from the following fact: Let Δ be an effective divisor on X and let Δ' be the proper transform of Δ by μ. Then $\mathrm{mult}_y(\Delta') \leq \mathrm{mult}_x(\Delta)$ for $y \in \mu^{-1}(x)$. Q.E.D.

Proposition (1.14). *If* $c_1(D) \in \mathrm{Mv}(X)$, *then there exist at most a finite number of subvarieties* W *of* X *with* $\sigma_W(D) > 0$ *and* $\mathrm{codim}\, W = 2$.

Proof. Let Z be the intersection of all the supports of effective $\mathbb{R}$-divisors which are numerically equivalent to D. Then by **(1.7,** (3)), $\operatorname{codim} Z \geq 2$. If $\sigma_W(D) > 0$, then $W \subset Z$. Q.E.D.

Proposition (1.15). *If* $\dim X = 3$, $N_\sigma(D) = 0$ *and* $D \cdot C_i < 0$ *for some finitely many irreducible curves* C_i, *then there exists a bimeromorphic morphism* $\pi : X \longrightarrow Z$ *onto a complex variety* Z *such that* $\pi(C_i)$ *are points and* π *induces an isomorphism* $X \setminus \cup C_i \simeq Z \setminus \cup \pi(C_i)$.

Proof. We may assume that $c_1(D) \in \mathrm{Mv}(X)$ and that D is a Cartier divisor. By the proof of **(1.7** (3)), we have two effective Cartier divisors Δ_1 and Δ_2 such that $\dim(\Delta_1 \cap \Delta_2) = 1$ and $\Delta_i \in |mD|$ for some $m > 0$. Since $D \cdot C_i < 0$, by [N2, (1.4)], we can contract all the C_i. Q.E.D.

§2. ν-decomposition.

We introduce another decomposition for a pseudo-effective $\mathbb{R}$-divisor D. We remark that for the positive part of the σ-decomposition $P_\sigma(D)$ and for all the prime divisors Γ, $P_\sigma(D)|_\Gamma$ is pseudo-effective. So let us consider the set

$$\mathfrak{S} := \{\, \Delta \mid \text{an effective } \mathbb{R}\text{-divisor such that } (D - \Delta)|_\Gamma \text{ is pseudo-effective for all prime divisor } \Gamma \text{ on } X \,\}.$$

Then we have:

Lemma (2.1). *The divisor* $N_\nu(D) := \sum_\Gamma \inf\{\mathrm{mult}_\Gamma(\Delta) \mid \Delta \in \mathfrak{S}\}\Gamma$ *is also an element of* $\mathfrak{S}$.

Proof. For any prime divisor Γ and for any positive number ε , there is an effective $\mathbb{R}$-divisor $\Delta \in \mathfrak{S}$ such that $\delta := \mathrm{mult}_\Gamma(N_\nu(D)) - \mathrm{mult}_\Gamma(\Delta) \leq \varepsilon$. Thus $(D - N_\nu(D))|_\Gamma + \delta\Gamma|_\Gamma = (D - \Delta)|_\Gamma + (\Delta' - N_\nu(D)')|_\Gamma$ is pseudo-effective, where $\Delta' := \Delta - \mathrm{mult}_\Gamma(\Delta)\Gamma$ and $N_\nu(D)' := N_\nu(D) - \mathrm{mult}_\Gamma(N_\nu(D))\Gamma$. Therefore $N_\nu(D) \in \mathfrak{S}$. Q.E.D.

Since $N_\sigma(D) \in \mathfrak{S}$, we have $N_\nu(D) \leq N_\sigma(D)$. Especially, $P_\nu(D) := D - N_\nu(D)$ is a pseudo-effective $\mathbb{R}$-divisor.

Definition (2.2). The decomposition $D = P_\nu(D) + N_\nu(D)$ is called the <u>ν-decomposition</u> of D . $P_\nu(D)$ and $N_\nu(D)$ are called the <u>positive part</u> and the <u>negative part</u> of the ν-decomposition of D , respectively.

Remark (2.3). Let $NMv(X)$ be the cone generated by the first Chern classes of the pseudo-effective $\mathbb{R}$-divisors Δ such that $\Delta_{|\Gamma}$ are still pseudo-effective for any prime divisors Γ . Then it is easy to see that for pseudo-effective $\mathbb{R}$-divisors D , $c_1(P_\nu(D)) \in NMv(X)$ and that if $c_1(D - \Delta) \in NMv(X)$ for some effective $\mathbb{R}$-divisor Δ , then $\Delta \geq N_\nu(D)$.

Remark (2.4). We can calculate the ν-decomposition of given D as follows: Let $\mathfrak{D}_1 = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_{m_1}\}$ be the set of prime divisors Γ such that $D|_\Gamma$ is not pseudo-effective. If $\mathfrak{D}_1$ is empty, we stop here. Otherwise, the set

$$\mathfrak{T}_1 := \{ (r_i)_{1\le i\le m_1} \mid 0 \le r_j \in \mathbb{R} \text{ and } (D - \sum_{i=1}^{m_1} r_i\Gamma_i)|_{\Gamma_j} \text{ is pseudo-effective for all } j \}$$

is non-empty and we have $(t_i^{(1)}) \in \mathfrak{T}_1$, where $t_j^{(1)} := \inf\{ t \ge 0 \mid t = r_j \text{ for some } (r_i) \in \mathfrak{T}_1 \}$, by the same argument as in the proof of **(2.1)**. Here we set

$$D^{(1)} := D - \sum_{i=1}^{m_1} t_i^{(1)}\Gamma_i .$$

$D^{(1)}$ is also pseudo-effective. Next we consider the set $\mathfrak{D}_2 = \{\Gamma_{m_1+1}, \Gamma_{m_1+2}, \ldots, \Gamma_{m_2}\}$ consisting of all the prime divisors Γ such that $D^{(1)}|_\Gamma$ is not pseudo-effective. If $\mathfrak{D}_2$ is empty, we stop here. Otherwise let $\mathfrak{T}_2$ be the set $\{ (r_i)_{1\le i\le m_2} \mid 0 \le r_j \in \mathbb{R}$ and $(D - \sum_{i=1}^{m_2} r_i\Gamma_i)|_{\Gamma_j}$ is pseudo-effective for all $0 \le j \le m_2 \}$. Then $(t_i^{(2)}) \in \mathfrak{T}_2$, where $t_j := \inf\{t \ge 0 \mid t = r_j$ for some $(r_i) \in \mathfrak{T}_2 \}$ and we set $D^{(2)} := D - \sum_{i=1}^{m_1} t_i^{(2)}\Gamma_i$. Similarly, if we get the sets $\mathfrak{D}_k$, $\mathfrak{T}_k$ and the divisor $D^{(k+1)}$. Since the prime divisors contained in some $\mathfrak{D}_k$ are components of $N_\sigma(D)$, this process must be terminated. The last divisor $D^{(k)}$ should be the positive part $P_\nu(D)$ of the ν-decomposition of D .

Remark (2.5). (1) Zariski [Z] and Fujita [F1] have constructed the Zariski-decomposition on surfaces just by the same way as above. But in their case, $t_i^{(1)}$, $t_i^{(2)} \cdots$, are calculated by linear equations.

(2) If $P_\nu(D) \in \overline{Mv}(X)$, then this ν-decomposition is nothing but the σ-decomposition by (1.9) and (2.3).

(3) In general, $N_\sigma(D) \neq N_\nu(D)$. For example, let us consider the point blow-up $f : Y \longrightarrow X$. Then $N_\nu(f^*D) = f^*N_\nu(D)$, but $N_\sigma(f^*D) \neq f^*N_\sigma(D)$, if $\sigma_x(D) > 0$.

Proposition (2.6). *If* $\dim X = 3$, $N_\nu(D) = 0$ *and* $D \cdot C < 0$ *for an irreducible curve* C , *then there exists a proper bimeromorphic morphism* $\pi : X \longrightarrow Z$ *such that* $\pi(C)$ *is a point and* π *induces an isomorphism* $X \setminus C \simeq Z \setminus \pi(C)$.

Proof. We may assume that D is a big Cartier divisor. Let Δ be an element of $|mD|$ for some positive integer m . Since $\Delta \cdot C < 0$, there is a component Γ of Δ such that $\Gamma \cdot C < 0$. Thus $\Gamma \supset C$. By the assumption, $\Delta|_\Gamma$ is a pseudo-effective divisor on Γ . Therefore there exists an effective Cartier divisor E on Γ such that $(E \cdot C)_\Gamma < 0$. Let $\mathcal{I}$ be the defining ideal of E on X . Then we have the following exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-\Gamma) \otimes \mathcal{O}_C \longrightarrow \mathcal{I} \otimes \mathcal{O}_C \longrightarrow \mathcal{O}_\Gamma(-E) \otimes \mathcal{O}_C \longrightarrow 0 .$$

Therefore $\mathcal{J}\otimes\mathcal{O}_C$ is an ample vector bundle. Hence by the contraction criterion [N2, (1.4)], we can contract the curve C . Q.E.D.

§3. Relative version.

Let $\pi : X \longrightarrow S$ be a projective surjective morphism from a complex manifold X onto a complex variety S . Fix a point P on S.

Definition (3.1). An $\mathbb{R}$-divisor D on X is called *π-big over P*, if for a π-ample divisor A , there exist a positive number ε and an effective $\mathbb{R}$-divisor Δ such that $D - \varepsilon A = \Delta$ in $N^1(X/S;\ P)_{\mathbb{Q}}$ (cf.[N1 §4]). An $\mathbb{R}$-divisor D is called *π-pseudo-effective over P*, if $D + \varepsilon A$ is π-big over P for any positive number ε .

Now we shall define the relative σ-decomposition. Let Γ be a prime divisor on X with $\pi(\Gamma) \ni P$ and let B be an $\mathbb{R}$-divisor which is π-big over P . First of all, we define the following numbers:

$$\sigma_\Gamma(B;\ P)_{\mathbb{Q}} := \inf\{\mathrm{mult}_\Gamma(\Delta) \mid \Delta \geq 0,\ \Delta = B \ \text{ in } N^1(X/S;\ P)_{\mathbb{Q}}\},$$
$$\sigma_\Gamma(B;\ P) := \inf\{\mathrm{mult}_\Gamma(\Delta) \mid \Delta \geq 0,\ \Delta = B \ \text{ in } N^1(X/S;\ P)\}.$$

By the same argument as in (1.1), we have

$\sigma_\Gamma(B;\ P)_{\mathbb{Q}} = \lim_{\varepsilon\downarrow 0} \sigma_\Gamma(B+\varepsilon A;\ P)_{\mathbb{Q}}$ for any π-ample divisor A and

$\sigma_\Gamma(B; P)_{\mathbb{Q}} = \sigma_\Gamma(B; P)$. Let D be an $\mathbb{R}$-divisor on X which is π-pseudo-effective over P.

Proposition (3.2). *(1). The limit* $\lim_{\varepsilon \downarrow 0} \sigma_\Gamma(D + \varepsilon A; P)$ *(might be* $+\infty$*) does not depend on the choice of a* π*-ample divisor* A. *(In what follows, we denote the limit by* $\sigma_\Gamma(D; P)$*).*

(2). If one of the following conditions are satisfied, then $\sigma_\Gamma(D; P) < +\infty$:

(a) $\pi(\Gamma) = S$,

(b) $\mathrm{codim}_S(\pi(\Gamma)) = 1$,

(c) There exists an effective $\mathbb{R}$*-divisor* Δ *with* $\Delta = D$ *in* $N^1(X/S; P)$.

Proof. (1). This is derived from the same argument as in (1.2,(2)).

(2). (a). It is easily proved restricting D to "general" fibers of π. (c). Trivial. (b). We may assume that π is a projective fiber space and S is normal. Let $\Gamma_0 := \Gamma$, Γ_1, Γ_2, ..., Γ_ℓ be all the prime divisors on X with $\pi(\Gamma_i) = \pi(\Gamma)$. Then there exist positive integers a_i $(1 \le i \le \ell)$, a reflexive sheaf $\mathcal{L}$ of rank one on S and a Zariski open subset U of S such that $\mathcal{L}_{|U}$ is invertible, $\mathrm{codim}_S(S \setminus U) \ge 2$ and

$$\pi^*(\mathcal{L}_{|U}) \simeq \mathcal{O}_X\left(\sum_{i=0}^{\ell} a_i \Gamma_i\right)_{|\pi^{-1}(U)} .$$

By taking a blowing-up of X, we may assume that

$\pi^*\pi_*\mathcal{O}_X(\sum_{i=0}^{\ell} a_i\Gamma_i)$/torsion is invertible and isomorphic to $\mathcal{O}_X(\sum_{i=0}^{\ell} a_i\Gamma_i - E)$ for some effective divisor E on X with codim $\pi(E) \geq 2$. Since $\sum_{i=0}^{\ell} a_i\Gamma_i - E$ is π-nef, we have $\sigma_{\Gamma_j}(\sum_{i=0}^{\ell} a_i\Gamma_i; P) \leq \sigma_{\Gamma_j}(E; P) = 0$. Thus $\sigma_{\Gamma_j}(D; P) = 0$ for some Γ_j. For any $\varepsilon > 0$, $(D + \varepsilon A - \sum_{i=0}^{\ell} \sigma_{\Gamma_i}(D+\varepsilon A; P)\Gamma_i)|_{\Gamma_j}$ is $(\pi|_{\Gamma_j})$-pseudo-effective over P. Hence if $\pi(\Gamma_k \cap \Gamma_j) = \pi(\Gamma)$, then $\sigma_{\Gamma_k}(D; P) < +\infty$. Since π is a fiber space, we have $\sigma_{\Gamma}(D; P) < +\infty$.

Q.E.D.

If $\sigma_{\Gamma}(D; P) < +\infty$ for all prime divisor Γ with $\pi(\Gamma) \ni P$, then the similar results as in (1.3), (1.4) and (1.5) are obtained. Thus we can define the negative part $N_{\sigma}(D; P)$ of the relative σ-decomposition to be the effective $\mathbb{R}$-divisor $\sum \sigma_{\Gamma}(D; P)\Gamma$. Also we can define the relative ν-decomposition as in §2.

§4. Examples.

(I). Let $f : X \longrightarrow Z$ be a proper bimeromorphic morphism from a 3-dimensional complex manifold X such that the f-exceptional set is just a compact smooth curve C. This morphism f is called the contraction of C and C is called an exceptional curve in X (cf. [N2]). Let P be the point $f(C)$. Now we shall consider the relative Zariski-decomposition Problem. Since $N^1(X/Z; P)$ is one dimensional, we treat a line bundle $\mathcal{L}$ on X with $\mathcal{L}\cdot C < 0$. Clearly $N_\sigma(\mathcal{L}; P) = 0$. To get the Zariski-decomposition of $\mathcal{L}$, we must consider the blow-up along C. We use the notation of [N2 §2]. Let $\mu_1 : X_1 \longrightarrow X$ be the blowing-up along C, and let E_1 be the exceptional divisor $\mu_1^{-1}(C) \simeq \mathbb{P}_C(I_C/I_C^2)$, where I_C is the defining ideal of C in X.

Proposition (4.1). If the conormal bundle I_C/I_C^2 is semi-stable, then we obtain $N_\nu(\mu_1^*\mathcal{L}) = -2(\mathcal{L}\cdot C)/\deg(I_C/I_C^2)E_1$ and the positive part $P_\nu(\mu_1^*\mathcal{L})$ is nef over Z. Namely, the relative Zariski-decomposition for $\mathcal{L}$ exists.

Proof. We must know when the divisor $\Delta := (\mu_1^*\mathcal{L} - xE_1)_{|E_1}$ is pseudo-effective. Since I_C/I_C^2 is semi-stable, every effective divisors are nef and Δ is pseudo-effective, if $\Delta^2 \geq 0$ and $x > 0$ by [Mi (3.1)]. Thus $N_\nu(\mu_1^*\mathcal{L}) = -2(\mathcal{L}\cdot C)/\deg(I_C/I_C^2)E_1$. Q.E.D.

Assume that the conormal bundle I_C/I_C^2 is not semi-stable. Then we have an exact sequence : $0 \longrightarrow \mathcal{L}_0 \longrightarrow I_C/I_C^2 \longrightarrow \mathcal{M}_0 \longrightarrow 0$, where $\mathcal{L}_0$ and $\mathcal{M}_0$ are line bundles on C with $\deg(\mathcal{L}_0) > \deg(\mathcal{M}_0)$. Thus there is the negative section C_1 on the ruled surface E_1 such that $\mathcal{O}(C_1)\otimes\mathcal{O}_{C_1} \simeq \mathcal{M}_0\otimes\mathcal{L}_0^{-1}$.

Proposition (4.2). *If* $2\deg(\mathcal{M}_0) \geq \deg(\mathcal{L}_0)$, *we have the relative Zariski-decomposition of* $\mathcal{L}$ *over* Z.

Proof. Let $\mu_2 : X_2 \longrightarrow X_1$ be the blowing-up along C_1, E_2 the μ_2-exceptional divisor and let E_1' be the proper transform of E_1. By the exact sequence:

$$0 \longrightarrow \mathcal{O}(-E_1)\otimes\mathcal{O}_{C_1} \longrightarrow I_{C_1}/I_{C_1}^2 \longrightarrow \mathcal{O}_{C_1}\otimes\mathcal{O}_{E_1}(-C_1) \longrightarrow 0 ,$$

if $2\deg(\mathcal{M}_0) > \deg(\mathcal{L}_0)$, then $C_2 := E_1' \cap E_2$ is the negative section of E_2. If $2\deg(\mathcal{M}_0) = \deg(\mathcal{L}_0)$, then E_2 is the ruled surface over C associated with a semi-stable vector bundle $I_{C_1}/I_{C_1}^2$. Therefore by [N2 (2.4)], we obtain a proper modification $\varphi : Y \longrightarrow X_2$ such that

(1) $\varphi^{-1}(E_1' \cup E_2)$ is a union of ruled surfaces F_j $(1 \le j \le k)$ over C for some $k \ge 2$,

(2) there is just one surface F_k is a ruled surface associated to a semi-stable vector bundle on C, and that

(3) the negative section of the other surface F_j $(j < k)$ is the complete intersection of F_j and other F_i.

Therefore if $\Delta_{|F_j}$ is pseudo-effective for a divisor Δ on Y, then $\Delta_{|F_j}$ are nef. Thus the ν-decomposition of the pull-back of $\mathcal{L}$ gives the relative Zariski-decomposition. Q.E.D.

By the same argument as in (4.2), we have:

Proposition (4.3). *If there exist a vector bundle* $\mathbb{E}$ *of rank two on* C *and an open neighborhood* U *of the zero section of* $\mathbb{E}$ *such* *that* U *is isomorphic to* X *, then we can find the relative Zariski-decomposition of* $\mathcal{L}$.

Proposition (4.4). *If there exist two irreducible divisors* Δ_1 *and* Δ_2 *such that* $\Delta_1 \cdot C < 0$, $\Delta_2 \cdot C < 0$ *and* $\Delta_1 \cap \Delta_2 = C$ *, then we can find the Zariski-decomposition.*

Proof. Let m_1 and m_2 be positive integers such that $m_1(\Delta_1 \cdot C_1) = m_2(\Delta_2 \cdot C_2)$. Let $f : V \longrightarrow X$ be the normalization of the blowing-up of X along the ideal $J := \mathcal{O}_X(-m_1\Delta_1) + \mathcal{O}_X(-m_2\Delta_2)$. Since $J \otimes \mathcal{O}_C \simeq \mathcal{O}_C(-m_1\Delta_1) \oplus \mathcal{O}_C(-m_2\Delta_2)$, $f^{-1}(C)_{red} =: E$ is the ruled surface over C associated with the semi-stable vector bundle $J \otimes \mathcal{O}_C$. The divisor E is $\mathbb{Q}$-Cartier, since $f^*J/\text{torsion}$ is an invertible sheaf and isomorphic to $\mathcal{O}_V(-kE)$ for some k . Let us consider the ν-decomposition of $f^*\mathcal{L}$ on V . Since the divisor on E is pseudo-effective if and only if it is nef, the positive part of the ν-decomposition is nef. Q.E.D.

Remark (4.5). (1). There is an example where the assumption of (4.4) is not satisfied: Let $0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_C \longrightarrow 0$ be the non-trivial extension over an elliptic curve C and let $\mathbb{E}$ be the total space $\mathbb{V}(\mathcal{E} \otimes \mathcal{N})$ of $\mathcal{E} \otimes \mathcal{N}$, where $\mathcal{N}$ is a negative line bundle on C . Then the zero-section of $\mathbb{E}$ is an exceptional curve, but there exist no such effective prime divisors Δ_1, Δ_2 on any neighborhoods of the zero-section as in (4.4).

(2). If there is a proper bimeromorphic morphism $X' \longrightarrow Z$ which is isomorphic outside of P and is not isomorphic to the original f, then the assumption of (4.4) is satisfied. But the converse does not hold in general. For example, let $\mathbb{E}$ be the total space $\mathbb{V}(\mathcal{O}_C \oplus \mathcal{M})$ of $\mathcal{O}_C \oplus \mathcal{M}$ on an elliptic curve C such that $\mathcal{M}$ has degree zero but is not a torsion element in $\mathrm{Pic}(C)$. Then the relative Zariski-decomposition for a divisor L on X with $L \cdot C < 0$ exists by (4.3), but its positive part is not relatively semi-ample over Z. Thus it is impossible to obtain the morphism $X' \longrightarrow Z$ above.

(II). Let $\mathcal{E}$ be a locally free sheaf of rank r on a projective smooth curve C, $\pi : X := \mathbb{P}_C(\mathcal{E}) \longrightarrow C$ the associated projective bundle and let $\mathcal{O}_{\mathcal{E}}(1)$ be the tautological line bundle on X such that $\pi_* \mathcal{O}_{\mathcal{E}}(1) \simeq \mathcal{E}$. We shall consider the Zariski-decomposition Problem for pseudo-effective divisors on X. Let F be a fiber of π. Then every $\mathbb{R}$-divisor is numerically equivalent to a linear combination of $\mathcal{O}_{\mathcal{E}}(1)$ and F. By [HN], we have a unique filtration (which is called the Harder-Narasimhan filtration of $\mathcal{E}$) by subbundles: $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_\ell = \mathcal{E}$, which satisfies the following two conditions:

(1) $\mathcal{E}_i / \mathcal{E}_{i-1}$ is a nonzero semi-stable vector bundle for all $1 \le i \le \ell$.

(2) $\mu(\mathcal{E}_i / \mathcal{E}_{i-1}) > \mu(\mathcal{E}_{i+1} / \mathcal{E}_i)$ for $1 \le i \le \ell - 1$, where $\mu(\mathcal{E}) := \deg(\mathcal{E}) / \mathrm{rank}(\mathcal{E})$.

Lemma (4.6). *For a real number* t, *the divisor* $\mathcal{O}_{\mathcal{E}}(1) - tF$ *is pseudo-effective if and only if* $t \le \mu(\mathcal{E}_1)$. *Also the divisor*

$\mathcal{O}_{\mathcal{E}}(1) - tF$ *is nef if and only if* $t \le \mu(\mathcal{E}/\mathcal{E}_{\ell-1})$.

Proof. First of all, we must mention that in the case of semi-stable $\mathcal{E}$, this lemma is proved in [Mi, (3.1)]. Therefore if $\mathbb{Q} \ni t < \mu(\mathcal{E}_1)$, then the divisor $\mathcal{O}_{\mathcal{E}_1}(1) - tF$ on $\mathbb{P}_C(\mathcal{E}_1)$ is big. Thus $h^0(C, \mathrm{Sym}^k(\mathcal{E}_1)\otimes\mathcal{A}^{tk}) \neq 0$ for $k >> 0$ with $tk \in \mathbb{Z}$, where $\mathcal{A}$ is a line bundle of degree one on C . Hence $h^0(C, \mathrm{Sym}^k(\mathcal{E})\otimes\mathcal{A}^{tk}) \neq 0$. Therefore $\mathcal{O}_{\mathcal{E}}(1) - tF$ is pseudo-effective for $t \le \mu(\mathcal{E}_1)$. Conversely, assume that $\mathbb{Q} \ni t > \mu(\mathcal{E}_1)$. Then the divisors $\mathcal{O}_{\mathcal{E}_i/\mathcal{E}_{i-1}}(1) - tF$ are not pseudo-effective on $\mathbb{P}_C(\mathcal{E}_i/\mathcal{E}_{i-1})$ for $1 \le i \le \ell$, respectively. From the Harder-Narasimhan filtration, we see that $h^0(C, \mathrm{Sym}^k(\mathcal{E})\otimes\mathcal{A}^{tk}) = 0$ for all $k > 0$. Thus $\mathcal{O}_{\mathcal{E}}(1) - tF$ is not pseudo-effective for $t > \mu(\mathcal{E}_1)$. Next we assume that $\mathcal{O}_{\mathcal{E}}(1) - tF$ is nef on X . Then $\mathcal{O}_{\mathcal{E}/\mathcal{E}_{\ell-1}}(1) - tF$ on $\mathbb{P}_C(\mathcal{E}/\mathcal{E}_{\ell-1}) \subset \mathbb{P}_C(\mathcal{E})$ is also nef. Therefore $t \le \mu(\mathcal{E}/\mathcal{E}_{\ell-1})$. Conversely, suppose that $\mathcal{O}_{\mathcal{E}}(1) - tF$ is not nef. Then there is an irreducible curve Γ in X with $(\mathcal{O}_{\mathcal{E}}(1) - tF)\cdot\Gamma < 0$. Let $C' \longrightarrow \Gamma$ be the normalization of Γ and let $f : C' \longrightarrow C$ be the composition with $\Gamma \longrightarrow C$. Since the Harder-Narasimhan filtration of $f^*\mathcal{E}$ is just the pull-back of $\mathcal{E}_i$'s , we may assume that Γ is a section of $X \longrightarrow C$. Then Γ corresponds to a surjective homomorphism $\mathcal{E} \longrightarrow \mathcal{F}$ onto an invertible sheaf $\mathcal{F}$ such that $\deg(\mathcal{F}) < t$. By a property of the Harder-Narasimhan filtration, we have $\mu(\mathcal{E}/\mathcal{E}_{\ell-1}) \le \deg(\mathcal{F})$. Therefore $t > \mu(\mathcal{E}/\mathcal{E}_{\ell-1})$. Q.E.D.

Proposition (4.7). *We have the Zariski-decomposition of the divisor*

$\mathcal{O}_{\mathcal{E}}(1) - \mu(\mathcal{E}_1)F$.

Proof. If $\mathcal{E}$ is semi-stable, we have nothing to prove. So we assume that $\ell \geq 2$. Let $f : Y \longrightarrow X$ be the blowing-up along $\mathbb{P}_C(\mathcal{E}/\mathcal{E}_1)$. Then we have a projective bundle structure $\pi : Y \longrightarrow \mathbb{P}_C(\mathcal{E}_1)$ such that the exceptional divisor E of f is isomorphic to the fiber product $\mathbb{P}_C(\mathcal{E}_1) \times_C \mathbb{P}_C(\mathcal{E}/\mathcal{E}_1)$, where the restrictions of π and f to E correspond to the first and second projections, respectively. We need the following:

Claim (4.8). *The* $\mathbb{R}$*-divisor* $\Delta = x\mathrm{pr}_1^*\mathcal{O}_{\mathcal{E}_1}(1) + y\mathrm{pr}_2^*\mathcal{O}_{\mathcal{E}/\mathcal{E}_1}(1) - tF$ *on* $\mathbb{P}_C(\mathcal{E}_1) \times_C \mathbb{P}_C(\mathcal{E}/\mathcal{E}_1)$ *is pseudo-effective if and only if* $x \geq 0$, $y \geq 0$ *and* $t \leq x\mu(\mathcal{E}_1) + y\mu(\mathcal{E}_2/\mathcal{E}_1)$.

Proof. First assume that $x,\ y \geq 0$ and $\lambda := x\mu(\mathcal{E}_1) + y\mu(\mathcal{E}_2/\mathcal{E}_1) - t \geq 0$. Then $\Delta = x\mathrm{pr}_1^*(\mathcal{O}_{\mathcal{E}}(1) - \mu(\mathcal{E}_1)F) + y\mathrm{pr}_2^*(\mathcal{O}_{\mathcal{E}/\mathcal{E}_1}(1) - \mu(\mathcal{E}_2/\mathcal{E}_1)F) + \lambda F$ is pseudo-effective by (4.6). Next suppose that Δ is pseudo-effective. Since $\mathcal{O}_{\mathcal{E}_1}(1) - \mu(\mathcal{E}_1)F$ is nef on $\mathbb{P}_C(\mathcal{E}_1)$, $H_\delta := \mathcal{O}_{\mathcal{E}_1}(1) - (\mu(\mathcal{E}_1) - \delta)F$ is ample for any positive rational number δ . Take a positive integer m so large that mH_δ is a very ample Cartier divisor and take general members $H_i \in |mH_\delta|$ for $1 \leq i \leq \mathrm{rank}(\mathcal{E}_1) - 1$ such that the intersection $Z := \cap H_i$ is a smooth curve on X . Then the restriction of Δ to the fiber product $Z \times_C \mathbb{P}_C(\mathcal{E}/\mathcal{E}_1) \simeq \mathbb{P}_Z(\tau^*(\mathcal{E}/\mathcal{E}_1))$ is still pseudo-effective, where τ is the morphism $Z \longrightarrow C$. Now this restricted divisor is numerically equivalent to:

$$y(\mathcal{O}_{\tau^*(\mathcal{E}/\mathcal{E}_1)}(1) - \mu(\tau^*(\mathcal{E}_2/\mathcal{E}_1))f) + \{x(\mathcal{O}_{\mathcal{E}_1}(1) - \mu(\mathcal{E}_1)F)\cdot Z + \lambda F\cdot Z\}f ,$$

where f denotes the fiber class of $\mathbb{P}_Z(\tau^*(\mathcal{E}/\mathcal{E}_1)) \longrightarrow Z$. Thus by (4.6), $y \geq 0$ and $x(\mathcal{O}_{\mathcal{E}_1}(1) - \mu(\mathcal{E}_1)F)\cdot Z + \lambda F\cdot Z \geq 0$. Hence $x(\mathcal{O}_{\mathcal{E}_1}(1) - \mu(\mathcal{E}_1)F)\cdot H_\delta^{\mathrm{rank}(\mathcal{E}_1)-1} + \lambda F\cdot H_\delta^{\mathrm{rank}(\mathcal{E}_1)-1} \geq 0$. Taking $\delta \downarrow 0$, we have $\lambda \geq 0$. Similarly, if we take a general ample intersection on $\mathbb{P}_C(\mathcal{E}/\mathcal{E}_1)$, then the inequality $x \geq 0$ is derived. Q.E.D.

Proof of (4.7) *continued:* We shall calculate the ν-decomposition of $f^*(\mathcal{O}_{\mathcal{E}}(1) - \mu(\mathcal{E}_1)F)$. Since $\pi^*\mathcal{O}_{\mathcal{E}_1}(1) \simeq f^*\mathcal{O}_{\mathcal{E}}(1)\otimes\mathcal{O}_Y(-E)$, $\mathcal{O}_E(-E) \simeq pr_1^*\mathcal{O}_{\mathcal{E}_1}(1) \otimes pr_2^*\mathcal{O}_{\mathcal{E}/\mathcal{E}_1}(-1)$. Therefore by (4.8), $(f^*(\mathcal{O}_{\mathcal{E}}(1) - \mu(\mathcal{E}_1)F) - \alpha E)_{|E}$ is pseudo-effective if and only if $0 \leq \alpha \leq 1$ and $\mu(\mathcal{E}_1) \leq \alpha\mu(\mathcal{E}_1) + (1-\alpha)\mu(\mathcal{E}_2/\mathcal{E}_1)$. Since $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2/\mathcal{E}_1)$, these inequalities hold if and only if $\alpha = 1$. Therefore $P_\nu(f^*(\mathcal{O}_{\mathcal{E}}(1) - \mu(\mathcal{E}_1)F)) = \pi^*(\mathcal{O}_{\mathcal{E}_1}(1) - \mu(\mathcal{E}_1)F)$, which is nef. Thus this is the Zariski-decomposition. Q.E.D.

Also by the same method as above, we can prove the following:

Proposition (4.9). *If* $\ell \leq 2$, *then the Zariski-decomposition exist for every pseudo-effective divisors on* X.

References.

[C] S. D. Cutkosky, Zariski decomposition of divisors on algebraic varieties, Duke Math. J., **53** (1986), 149 - 156.

[F1] T. Fujita, On Zariski problem, Proc. Japan Acad. Ser. A, **55** (1979), 106 - 110.

[F2] ——, Zariski decomposition and canonical rings of elliptic threefolds, J. Math. Soc. Japan, **38** (1986), 19 - 37.

[HN] G. Harder and M. S. Narasimhan, On the cohomology groups of moduli space of vector bundles on curves, Math. Ann., **212** (1975), 215 - 248.

[K1] Y. Kawamata, The Zariski decomposition of log-canonical divisors, in Algebraic Geometry, Bowdoin, 1985, Proc. of Symp. in Pure Math. vol. **46** (1987), 425 - 433.

[K2] ——, Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces, Ann. of Math. **127** (1988), 93 - 163.

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the minimal model problem, in Algebraic Geometry, Sendai, 1985, Adv. Stud. in Pure Math. vol. **10** (1987), 283 - 360.

[Ma] K. Matsuda, On the numerically fixed parts of line bundles, Proc. Japan Acad. Ser. A, **61** (1985), 219 - 221.

[Mi] Y. Miyaoka, The Chern classes and the Kodaira dimension of a minimal variety, in Algebraic Geometry, Sendai, 1985, Adv. Stud. in Pure Math. vol. **10** (1987), 449 - 476.

[Mw] A. Moriwaki, Semi-ampleness of the numerically effective part of Zariski-decomposition, J. Math. Kyoto Univ., **26** (1986), 461 - 481.

[N1] N. Nakayama, The lower semi-continuity of the plurigenera of complex varieties, in Algebraic Geometry, Sendai, 1985, Adv. Stud. in Pure Math. vol. **10** (1987), 551 - 590.

[N2] ——, On smooth exceptional curves in threefolds, preprint 1988.

[Z] O. Zariski, The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface, Ann. of Math., **76** (1962), 560 - 615.

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Rational curves on Weierstrass models

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§1. Introduction

Recently, a lot of examples of Calabi-Yau threefolds(i.e. compact Kähler threefolds with trivial first Chern class and with finite fundamental group) have been constructed. In contrast with the case of $K3$ surfaces, it is observed that they have a large repertory of Euler numbers, in other words, they can not be connected by deformation in the proper sense. But some families of Calabi-Yau threefolds can be regarded as strata of other families by means of small resolutions. The following result was shown by H. Clemens and R. Friedman in [1],[2]: *A general quintic threefold has many mutually disjoint $(-1,-1)$-curves(i.e. rational curves with normal bundles isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$) and the nodal variety obtained by the contraction of at least two of them is deformed to a 2-connected complex manifold V with $K_V = 0$.* By the result of C.T.C. Wall [11] such a complex manifold V is diffeomorphic to a conneted sum of N copies of $S^3 \times S^3$, $(S^3 \times S^3)^{\# N}$, where $N = 1/2\ b_3(V)$. Prompted by this result, M. Reid has suggested in [9] that all families of Calabi-Yau threefolds with $\pi_1 = 0$ may be embedded in an irreducible moduli space as strata, and that a general point of this moduli space corresponds to a non-Kähler 2-connected threefold. For a more general and detailed statement, see [9].

In this paper we restrict our attention to the smooth projective threefolds X with the follwing properties (*)

(*) 1) X has an elliptic fibration with a rational section over a smooth rational surface S.

2) $P_g(X) = 1$ and $\kappa(X) = 0$.

Remark that many Calabi-Yau threefolds are contained in the above class. For example, the Calabi-Yau threefolds studied in [10] satisfy (*).

We want to study the relationship between these X(exacxtly speaking, their birational classes) and 2-connected non-Kähler threefolds with trivial canonical bundle. My plan to this problem at this point (which might need some modification, and only one part of Step 1 has been completed) is divided into two steps:

Step 1. For X with the properties (*), construct a flat family of threefolds $f: \mathscr{X} \longrightarrow \Delta$: unit disk , such that

1) the central fibre $\mathscr{X}_0$ is a (possibly singular) projective threefold birational to X,

2) a general fibre $\mathscr{X}_t$ is a smooth, simply-connected, projective threefold with trivial canonical line bundle,

3) there are sub-flat families $\mathscr{C}_i$ $(1 \leq i \leq m)$ of curves of $\mathscr{X}$,

$$\begin{tikzcd} \mathscr{C}_i \arrow[r, phantom, "\subset"] \arrow[dr, "\text{flat}"'] & \mathscr{X} \arrow[d] \\ & \Delta \end{tikzcd}$$
, where $\mathscr{C}_{i,t}$ is a curve on $\mathscr{X}_t$

4) $\mathscr{C}_i$ and $\mathscr{C}_j$ $(i \neq j)$ do not intersect over $\Delta^* = \Delta - \{0\}$

5) For $t \neq 0$, $\mathscr{C}_{i,t}$ $(1 \leq i \leq m)$ are mutually disjoint $(-1,-1)$-curves on $\mathscr{X}_t$(i.e. smooth rational curves with

normal bundles $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$),

6) For $t \neq 0$, the complex analytic space $\bar{\mathfrak{X}}_t$ obtained from $\mathfrak{X}_t$ by contracting $\mathcal{C}_{i,t}$'s , can be deformed to a non-Kähler 2-connected complex manifold with trivial canonical line bundle.

Step 2. Observation of the relationship between $\mathfrak{X}_0$ and the non-Kähler manifold: For example, it is desired that there is a semi-stable degeneration g: $\mathfrak{Y} \longrightarrow \Delta$ such that

1) one component of $\mathfrak{Y}_0$ is bimeromorphic to $\mathfrak{X}_0$(hence, to X itself), and

2) a general fibre $\mathfrak{Y}_t$ is a non-Kähler 2-connected complex manifold with trivial canonical line bundle.

The purpose of this paper is to introduce a partial result with respect to Step 1. Our starting point towards Step 1 are the following , one of which is the result of N.Nakayama:

Lemma Let f:$\mathfrak{X} \longrightarrow T$ be a flat projective morphism between algebraic varieties. Assume that a general fibre of f is a smooth threefold. Let t_0 and t be two distinct point of T , and assume the follwing:

a) $\mathfrak{X}_{t_0}$ is a (possibly singular) projective variety.

b) $\mathfrak{X}_t$ is a smooth simply-connected threefold with trivial canonical line bundle.

c) For $\mathfrak{X}_t$, there are mutually disjoint (-1,-1)-curves $C_{i,t}$'s on $\mathfrak{X}_t$, and the complex analytic space $\bar{\mathfrak{X}}_t$ obtained by contracting these curves can be deformed to a non-Kähler 2-connected complex manifold with trivial canonical line bundle.

Then there is a flat family f: $\mathfrak{X} \longrightarrow \Delta$, for which $\mathfrak{X}_0 = \mathfrak{X}_{t_0}$ and the properties 1)... 6) in Step 1 are satisfied.

Proof

Let D be an irreducible curve on T passing through t and t_0. If necessary, taking its normalization, we have a flat family of threefolds $\mathfrak{X} \longrightarrow D$, where D is a smooth curve. $Hilb_{\mathfrak{X}/D}$ is étale over D at $[C_{i,t}]$ because $N_{C_{i,t}/\mathfrak{X}_t} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Let us denote by H_i the irreducible component of $Hilb_{\mathfrak{X}/D}$ containing $[C_{i,t}]$. For different i and j, H_i and H_j may possibly coincide. But by taking a suitable finite cover of D, we may assume that $H_i \neq H_j$ for i,j with $i \neq j$, and that each H_i is birational to D. In this situation, we restrict our attention to a sufficiently small neighbourhood Δ of t_0.

From now on, we use the following notation:

$$\begin{array}{ccccccc} \mathfrak{X} \times_D \Delta =: & \mathfrak{X}_\Delta & \subset & \mathfrak{X} & \qquad & H_i \times_D \Delta =: & H_{i,\Delta} \\ & \downarrow & & \downarrow & & & \downarrow p_i \\ t_0 = 0 \in & \Delta & \subset & D & & & \Delta \end{array}$$

p_i can be assumed to be an isomorphism over Δ^*, because H_i is birational to D. Denote by $\mathcal{H}_i$ the universal family over H_i. For a general point $h \in H_i$, $\mathcal{H}_i \times k(h)$ is isomorphic to $\mathbb{P}^1$. Therefore we may assume that $\mathcal{H}_{i,\Delta} := \mathcal{H}_i \times_D \Delta$ is a $\mathbb{P}^1$- bundle over H_{i,Δ^*}. Moreover we may assume that for a point $h \in H_{i,\Delta^*}$, $\mathcal{H}_i \times k(h)$ $(1 \leq i \leq m)$ are mutually disjoint $(-1,-1)$-curves on $\mathfrak{X}_{p_i(h)}$, since $C_{i,t}$ are mutually disjoint $(-1,-1)$-curves. Let $\mathcal{E}_i$ be the image (with reduced structure) of $\mathcal{H}_{i,\Delta}$ in $\mathfrak{X}_\Delta$. Then $\mathcal{E}_i$ is flat over Δ^* because $\mathcal{E}_i$ is irreducible, reduced and dominating 1-dimensional disk. By the construction, $\mathcal{E}_i \cap \mathcal{E}_j = \phi$ for i,j $(i \neq j)$ over Δ^*. Finally remark that 2) in Step 1. is a topological condition. As for 6) in Step 1. see [2].

Theorem ([6,(3.4)]) *Let X be a smooth projective threefold with* (*). *Then X is birational to a Weierstrass model W over $\mathbb{P}^2$ or Σ_r* ($0 \le r \le 12$).*(For the definition of a Weierstrass model, see* (2.11).) *Moreover W has only rational Gorenstein singularities.*

By the above results, we may only consider the Weierstrass models W over $\mathbb{P}^2$ or Σ_r ($0 \le r \le 12$). But the following lemma shows that the case of Σ_2 can be excluded in our context.

Lemma (Jumping of the complex structure Σ_r)

1) Let r be an even (resp. odd) integer.
   Then for Σ_r ($r \ge 2$), there is a smooth projective morphism $f_r\colon \mathcal{S} \longrightarrow C_r$ such that a) C_r is a smooth curve (not compact) with a distinguished point t_0 , b) $\mathcal{S}_{t_0}$ is isomorphic to Σ_r , and c) $\mathcal{S}_t$ ($t \ne t_0$) is isomorphic to Σ_0 (resp. Σ_1).
2) For an arbitrary n, $\chi(\mathcal{O}_{\Sigma_r}(-nK_{\Sigma_r})) = (2n+1)^2$.
   If $0 \le r \le 2$, then $h^1(\mathcal{O}_{\Sigma_r}(-nK_{\Sigma_r})) = h^2(\mathcal{O}_{\Sigma_r}(-nK_{\Sigma_r})) = 0$ for $n \ge 1$.

Proof

1) See [12] p.205 , Step M).
2) By the theorem of Riemann-Roch, $\chi(\mathcal{O}_{\Sigma_r}(-nK_{\Sigma_r})) = (2n+1)^2$.

By the Serre duality, we have

$$h^2(\mathcal{O}_{\Sigma_r}(-nK_{\Sigma_r})) = h^0(\mathcal{O}_{\Sigma_r}((n+1)K_{\Sigma_r})),$$
$$h^1(\mathcal{O}_{\Sigma_r}(-nK_{\Sigma_r})) = h^1(\mathcal{O}_{\Sigma_r}((n+1)K_{\Sigma_r})).$$

We shall prove by induction that the right hand sides are zero for $n \ge 1$.

Since, for n = 1, $-K \sim 2C_0 + (2 + r)f$ (C_0: negative section , f: fibre), we can find a reduced element D in $|-K|$. Indeed, take as D, $C_0 + C + 2f$, where C is a section of self-intersection number r. From the exact sequence

$$0 \longrightarrow \mathcal{O}(K) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

we have

$$\begin{array}{ccccccc} H^0(\mathcal{O}) & \longrightarrow & H^0(\mathcal{O}_D) & \longrightarrow & H^1(\mathcal{O}(K)) & \longrightarrow & H^1(\mathcal{O}) = 0 . \\ \| & & \| & & & & \\ \mathbb{C} & \simeq & \mathbb{C} & & & & \end{array}$$

This shows that $H^1(\mathcal{O}(K)) = 0$. For the case n $\geq$ 2, from the exact sequence

$$0 \longrightarrow \mathcal{O}(nK) \longrightarrow \mathcal{O}((n-1)K) \longrightarrow \mathcal{O}_D((n-1)K) \longrightarrow 0,$$

we have

$$H^0(\mathcal{O}_D((n-1)K) \longrightarrow H^1(\mathcal{O}(nK)) \longrightarrow H^1(\mathcal{O}((n-1)K)) = 0 \text{ (by induction).}$$

Since $D = C_0 + C + 2f$, $(C_0 .(n-1)K) = (n-1)(r-2)$, $(C. (n-1)K) < 0$, $(f. (n-1)K) < 0$, and $(C_0 . 2f) = 2$, in the cases r $\leq$ 2 $H^0(\mathcal{O}_D((n-1)K) = 0$, which implies $H^1(\mathcal{O}(nK)) = 0$. Q.E.D.

Let W_2 be a given Weierstrass model over Σ_2, and W_0 a smooth Weierstrass model over Σ_0. Then , by the above lemma, we can connect two varieties by a flat deformation , and a general member of this flat deformation is a smooth Weierstrass model over Σ_0. Indeed, we may consider the following flat morphism(with the notation in Lemma) :

$$\begin{array}{ccc} \mathbb{W} & \supset & W(K_{\mathcal{G}_t}, a(t), b(t)) \\ \downarrow & & \downarrow \\ \mathbb{V}_{C_2}(R^0 f_{2*}\omega_{\mathcal{G}/C_2}^{\otimes -4}) \times \mathbb{V}_{C_2}(R^0 f_{2*}\omega_{\mathcal{G}/C_2}^{\otimes -6}) & \ni & (a(t), b(t)) \end{array}$$

,where $t \in C_2$,

$a(t) \in H^0(\omega_{\mathcal{G}_t}^{\otimes -4})$,

$b(t) \in H^0(\omega_{\mathcal{G}_t}^{\otimes -6})$,

$4a^3 + 27b^2$ not identically

zero.

Therefore we may only consider the Weierstrass models over $\mathbb{P}^2$, Σ_r $(0 \le r \le 12,\ r \ne 2)$. We are now in a position to state the main result of this paper.

Theorem

Let W be a general smooth Weierstrass model over $\mathbb{P}^1 \times \mathbb{P}^1$. Then there are mutually disjoint rational curves $C_1, \cdots, C_4$ on W with the following properties:

1) $N_{C_i/W} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ for each i.

2) $C_1, \cdots, C_4$ span $H_2(W\ ;\ \mathbb{Z})$.

Moreover, if we denote by $\varphi : W \longrightarrow \hat{W}$ the contraction of C_i's, then $\hat{W}$ can be deformed to a 2-connected compact complex manifold with $K = 0$.

The infinitesimal deformation spaces of Weierstrass models are very large. For example, $h^1(W, \Theta_W) = 243$ in our case where W is a Weierstrass model over $\mathbb{P}^1 \times \mathbb{P}^1$.

We shall sketch the proof of the above result. First we relate a singular Weierstrass model W over $\mathbb{P}^1 \times \mathbb{P}^1$ with a fibre product $X = S \times_{\mathbb{P}^1} S^+$ of two rational elliptic surface with sections. The birational map between W and X is a composition of flops and divisorial contractions. The merit of considering X is the point that it is easy to study the rational curves on it. Next, we choose suitable isolated rational curves on X. It is desired that α) the birational map between X and W does not make any effect on thses curves, and β) these curves have different numerical classes. We need 2) to proceed with the argument along the idea [2]. Then, what kind of curves should we find on X ? We note that S has many

representations as blow-ups of $\mathbb{P}^1 \times \mathbb{P}^1$. Take one of such representations and consider the linear system Σ on S coming from the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$. A general member C of Σ is a nonsingular rational curve and is a double cover of $\mathbb{P}^1$ by π: $S \longrightarrow \mathbb{P}^1$. By Hurwitz formula, C has two ramification points $\{P_1, P_2\}$. Also for λ: $S^+ \longrightarrow \mathbb{P}^1$, we consider similarly $C^+ \in \Sigma^+$ and $\{Q_1, Q_2\}$. If $\pi(P_1) = \lambda(Q_1)$ and $\pi(P_2) = \lambda(Q_2)$, then $C \times_{\mathbb{P}^1} C^+ \subset X$ consits of two nonsingular rational curves D_1 and D_2 intersecting at two points transversely. Under the suitable conditions (which can be described by using the degree of a *ramification map* (Def.(2.5))) , D_i is shown to be isolated. (Prop.(2.10)) The main object of this paper is a curve of such a type. Changing the representation of S(resp. S^+) as a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$, and replacing Σ (resp. Σ^+) by new one, we can find many isolated rational curves of different numerical types. We must take these curves mutually disjoint. For this purpose, in §3, we observe the degeneration of D_i. As a consequence, we will be able to choose suitable curves for which α) and β) holds. These curves are regarded as the curves on W. The normal bundles of these curves in W are $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $\mathcal{O} \oplus \mathcal{O}(-2)$. In the latter case, we must deform the curve and W so that it splits up into a union of curves of the former type. Thus §4 deals with the deformation of rational curves on W.(Prop.(4.4)) In Appendix, we calculate $H_2(W ; \mathbb{Z})$ for a nonsingular Weierstrass model W and show that it is torsion-free , which will guarantee 2-connectedness of the smoothing of $\widetilde{W}$ in Theorem of §4.
In the case where W is a Weierstrass models over $\mathbb{P}^2$, it seems that we must deal with the curves C of (1,-3) type(i.e. $N_{C/W} \simeq \mathcal{O}(1) \oplus \mathcal{O}(-3)$) and their deformation.

§ 2. Preliminaries

Let S and S^+ be rational elliptic surfaces with sections, and we denote by π and λ, their fibrations over $\mathbb{P}^1$, respectively. We shall consider the case where the following are satisfied:

(2.1) Generic fibres of π and λ are not isogenous to each other.

(2.2) All singular fibres of π and λ are irreducible.

Taking the fibre product of S and S^+ over $\mathbb{P}^1$, we obtain the the following diagram (2.3):

$$\begin{tikzcd} & S \times_{\mathbb{P}^1} S^+ \arrow[dl, "p"'] \arrow[dd, "f"] \arrow[dr, "q"] & \\ S \arrow[dr, "\pi"'] & & S^+ \arrow[dl, "\lambda"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

From the assumption(2.2) we have the following commutative exact diagram:

$$(2.4)\qquad \begin{tikzcd} & \mathrm{Pic}\, S \times \mathrm{Pic}\, S^+ \arrow[r] \arrow[d, "p^* \otimes q^*"] & \mathrm{Pic}\, \pi^{-1}(\eta) \times \mathrm{Pic}\, \lambda^{-1}(\eta) \arrow[r] \arrow[d, "p^*(\eta) \otimes q^*(\eta)"] & 0 \\ \mathbb{Z}\, f^*\mathcal{O}(1) \arrow[r] & \mathrm{Pic}\, S \times_{\mathbb{P}^1} S^+ \arrow[r] & \mathrm{Pic}\, f^{-1}(\eta) \arrow[r] & 0, \end{tikzcd}$$

where η is a generic point of $\mathbb{P}^1$. $p^*(\eta) \otimes q^*(\eta)$ is an isomorphism by (2.1). Since $f^*\mathcal{O}(1) \simeq p^*\pi^*\mathcal{O}(1)$, $p^* \otimes q^*$ is surjective. Hence for a line bundle $\mathfrak{F}$ on X, we have $\mathfrak{F} = p^*L \otimes q^*M$, where L (or M) is a line bundle on S (or S^+, resp). By Künneth formula, $f_*(\mathfrak{F}) \simeq \pi_*L \otimes \lambda_*M$, and if $\mathfrak{F}$ is effective, we may assume that L and M are effective. In fact, if $\mathfrak{F}$ is effective, then $f_*\mathfrak{F}$ has a non-zero section. Since π_*L and λ_*M are vector bundles on $\mathbb{P}^1$, they are direct sums of line bundles. Hence we conclude that for a suitable line bundle K on $\mathbb{P}^1$, $\pi_*L \otimes K$ and $\lambda_*M \otimes K^{-1}$ have non-zero sections. We can write $\mathfrak{F} \simeq p^*(L \otimes \pi^*K) \otimes q^*(M \otimes \lambda^*K^{-1})$. Since $L \otimes \pi^*K$ and $M \otimes \lambda^*K^{-1}$ are effective, we may assume that L and M are effective.

Remark Set $X = S \times_{\mathbb{P}^1} S^+$. Then X has at worst terminal singularities because the singular fibres of π and λ are of type I_1 or II. Moreover X is $\mathbb{Q}$-factorial(i.e. for any Weil divisor D on X, some multiple mD is a Cartier divisor). This is shown as follows: If X is not $\mathbb{Q}$-factorial, then there is a small projective resolution $r: Y \longrightarrow X$ for which Y is $\mathbb{Q}$-factorial. But then we have (2.4) for Y instead of X and the exceptional curve of r is numerically equivalent to 0, which contradicts the projectivity of Y. Hence X is $\mathbb{Q}$-factorial. On the other hand, since X is a Gorenstein 3-fold, X is factorial by [4, §5].

Definition(2.5) Let $\pi: S \longrightarrow \mathbb{P}^1$ be a rational elliptic surface with sections. Let Σ be a linear system on S whose general member C is a nonsingular rational curve with $C^2 = 0$. Then the *ramification map* $\varphi_{(S,\pi,\Sigma)}$ is defined as follows.

The members of Σ is parametrized by a projective line $\mathbb{P}^1$, because dim $\Sigma = 1$. We shall denote by C_t, the member of Σ corresponding to $t \in \mathbb{P}^1$. For a general $t \in \mathbb{P}^1$, $\pi|_{C_t}: C_t \longrightarrow \mathbb{P}^1$ is a double cover with two ramification point P_t and Q_t. This correspondence from t to $\{P_t, Q_t\}$ is extended to the map $\varphi_{(S,\pi,\Sigma)}: \mathbb{P}^1 \longrightarrow \mathcal{S}$, where $\mathcal{S}$ is a symmetric product of two projective lines. We denote $[\,\mathbb{C}(\mathbb{P}^1): \mathbb{C}(Im\ \varphi_{(S,\pi,\Sigma)})]$ by deg $\varphi_{(S,\pi,\Sigma)}$.

From now, we shall consider the case where S is obtained by blowing-ups of $\mathbb{P}^1 \times \mathbb{P}^1$ and Σ is the pull back of $|p_1{*}\mathcal{O}(1)|$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

For example, if we assume that every singular fibre is irreducible, then we always have this case.

Let us identify the vector space $H^0(\mathbb{P}^1 \times \mathbb{P}^1,\ p_1^*\mathcal{O}(2)\otimes p_2^*\mathcal{O}(2))$ with $\mathbb{C}^9$.

Let $T := \{ (x,y) \in \mathbb{C}^9 \times \mathbb{C}^9\ ;\ x = \alpha y$ or $y = \alpha x$ for some constant $\alpha \in \mathbb{C} \}$. Then T is a closed subset of $\mathbb{C}^9 \times \mathbb{C}^9$. Consider the $\mathbb{C}^*$-action on $\mathbb{C}^9 \times \mathbb{C}^9$ such that for $\gamma \in \mathbb{C}^*$, $\gamma(x,y) = (\gamma x, \gamma y)$. Then T is invariant under this action, and hence we have the following quotient spaces with respect to this action.

$$\begin{array}{ccc} \mathbb{C}^9 \times \mathbb{C}^9 & \longrightarrow & \mathbb{P}^{17} \\ \cup & & \cup \\ \mathbb{C}^9 \times \mathbb{C}^9 - T & \longrightarrow & U \end{array}$$

By the identification of $\mathbb{C}^9$ with $H^0(\mathbb{P}^1 \times \mathbb{P}^1,\ p_1^*\mathcal{O}(2)\otimes p_2^*\mathcal{O}(2))$, a point $(x,y) \in \mathbb{C}^9 \times \mathbb{C}^9 - T$ corresponds to a pair (f,g) of sections of $p_1^*\mathcal{O}(2)\otimes p_2^*\mathcal{O}(2)$. Since (x,y) is not contained in T, (f,g) determines a linear pencil $\lambda f + \mu g$; $(\lambda:\mu) \in \mathbb{P}^1$. Thus we have

$$\begin{array}{ccc} (\lambda:\mu\ , [(f,g)]) & \in & \mathbb{P}^1 \times U \\ \downarrow & & \downarrow \\ [(f,g)] & \in & U\ . \end{array}$$

If we replace U by a suitable open set of U, then each linear pencil $\mathbb{P}^1 \times \{u\}$; $u \in U$ has eight base points in number, and the blow-up of these points is a rational elliptic surface with sections S_u. For this S_u and the linear pencil Σ (the pull back of $|p_1^*\mathcal{O}(1)|$ on $\mathbb{P}^1 \times \mathbb{P}^1$), the ramification map φ_u is defined. On a suitable open set $V \subset U$, there is a commutative diagram:

$$
\begin{tikzcd}
\mathbb{P}^1 \times V \arrow[rr, "\varphi"] \arrow[dr] & & \mathcal{G} \times V \arrow[dl] \\
& V &
\end{tikzcd}
$$

such that $\varphi|_u = \varphi_u$.

We can find, by (2.6) below, at least one point $u_0 \in V$ such that $\deg \varphi_{u_0} = 1$, which shows that for a general point $u \in V$, $\deg \varphi_u = 1$. Indeed, since $\deg \varphi_{u_0} = 1$, for a general point x on $\mathbb{P}^1 \times \{u_0\}$ φ_{u_0} is a closed immersion at x. On the other hand, φ is a finite morphism at x, which implies that φ is a closed immersion at x by Nakayama's lemma. This shows that for any point $u \in V$ near u_0, $\deg \varphi_u = 1$.

(2.6) Explicit description of φ

$(T_0:T_1) \times (S_0:S_1)$:= a bi-homogenous coordinate of $\mathbb{P}^1 \times \mathbb{P}^1$

$$
\begin{aligned}
f = {} & a_1T_0^2S_0^2 + a_2T_0^2S_0S_1 + a_3T_0^2S_1^2 + a_4T_0T_1S_0^2 + a_5T_0T_1S_0S_1 + \\
& a_6T_0T_1S_1^2 + a_7T_1^2S_0^2 + a_8T_1^2S_0S_1 + a_9T_1^2S_1^2 \\
g = {} & b_1T_0^2S_0^2 + b_2T_0^2S_0S_1 + b_3T_0^2S_1^2 + b_4T_0T_1S_0^2 + b_5T_0T_1S_0S_1 + \\
& b_6T_0T_1S_1^2 + b_7T_1^2S_0^2 + b_8T_1^2S_0S_1 + b_9T_1^2S_1^2
\end{aligned}
$$

Consider the linear system $\lambda f + \mu g$ $(\lambda:\mu) \in \mathbb{P}^1$. If a_i's and b_j's are generally chosen, then this linear system has eight base points in number. Blowing up the base points, we have a rational elliptic surface with sections. Set $\Sigma = |p_2^*\mathcal{O}(1)|$. Then Σ is regarded as a linear system of the rational elliptic surface and we can consider the ramification map φ with respect to Σ. Σ is parametrized by $(S_0:S_1) \in \mathbb{P}^1$.

Put

$$
\begin{aligned}
F = {} & \{\lambda(a_7S_0^2 + a_8S_0S_1 + a_9S_1^2) + \mu(b_7S_0^2 + b_8S_0S_1 + b_9S_1^2)\}\, T_1^2 + \\
& \{\lambda(a_4S_0^2 + a_5S_0S_1 + a_6S_1^2) + \mu(b_4S_0^2 + b_5S_0S_1 + b_6S_1^2)\}\, T_1T_0 + \\
& \{\lambda(a_1S_0^2 + a_2S_0S_1 + a_3S_1^2) + \mu(b_1S_0^2 + b_2S_0S_1 + b_3S_1^2)\}\, T_0^2
\end{aligned}
$$

$$= \psi_1(\lambda, \mu, a, b, S_0, S_1)\, T_1^2 + \psi_2(\lambda, \mu, a, b, S_0, S_1)\, T_0T_1 + \psi_3(\lambda, \mu, a, b, S_0, S_1)\, T_0^2 .$$

Taking the discriminant $D(F)$, we write

$$D(F) = \psi_2^2 - 4\,\psi_1\psi_3$$
$$= \lambda^2\, \phi_1(a, S_0, S_1) + \lambda\mu\, \phi_2(a, b, S_0, S_1) + \mu^2\, \phi_3(b, S_0, S_1),$$

where ϕ_1, ϕ_2 and ϕ_3 are homogenous polynomials of deg 2 with respect to (a, b), and of deg 4 with respect to (S_0, S_1).

Then φ is defined as follows:

$$\begin{tikzcd}
(S_0:S_1)\times(a, b) \arrow[d, phantom, sloped, "\in"] \arrow[rr, dashed, "\varphi"] \arrow[dr] & & (\phi_1:\phi_2:\phi_3)\times(a, b) \arrow[d, phantom, sloped, "\in"] \arrow[dl] \\
\mathbb{P}^1\times\mathbb{P}^{17} \arrow[dr] & (a, b) \arrow[d, phantom, sloped, "\in"] & \mathbb{P}^2\times\mathbb{P}^{17} \arrow[dl] \\
& \mathbb{P}^{17} , &
\end{tikzcd}$$

where we identify the symmetric product $\mathcal{S}$ of two projective lines with $\mathbb{P}^2$. If three polynomials ϕ_1, ϕ_2 and ϕ_3 have common factors for a point $u_0 = (a_0, b_0) \in \mathbb{P}^{17}$, then φ is not a morphism above u_0 (φ_{u_0} can be regarded as a morphism fibrewisely). This is the reason why we shrink U to V in (2.5).

For example if we put $b_0 = (0, 0, 1/2, 1, 0, 1, 1, 0, 1/2)$ and $a_0 = (0, 1, 0, 0, 1, 1, 0, 1, 1)$, then φ is a morphism above $u_0 = (a_0, b_0) \in \mathbb{P}^{17}$. Moreover φ_{u_0} is described on $U_0 := (S_0 \neq 0)$ as follows:

$$\begin{array}{ccccc}
\varphi_{u_0} : (s) & \longrightarrow & (s^4 - 2s^3 + (\text{lower trems}), & & -2s^3 + (\text{lower terms})), \\
\| & & \| & & \| \\
S_1/S_0 & & W_0/W_2 & & W_1/W_2
\end{array}$$

where $(W_0:W_1:W_2)$ is a homogenous coordinate of $\mathbb{P}^2$. Writing $\alpha(s) = s^4 - 2s^3 + (\text{lower terms})$ and $\beta(s) = -2s^3 + (\text{lower terms})$, we have $\mathbb{C}(s) = \mathbb{C}(\alpha(s), \beta(s))$. Hence we have $\deg \varphi_{u_0} = 1$.

By (2.5) and (2.6) we have

Lemma(2.7) *For a general triplet* (S,π,Σ), $\deg \varphi_{(S,\pi,\Sigma)} = 1$.

Remark(2.8) Since S is a rational elliptic surface with sections, π is detemined uniquely up to $Aut(\mathbb{P}^1)$ for a fixed S. Thus the above property is independent of the choice of π if we fix S and Σ.

(2.9) Let (S, Σ, π) and (S^+, Σ^+, λ) be the same as above. We assume (2.1) and (2.2). We employ the following notation.

$X := S \times_{\mathbb{P}^1} S^+$

$\alpha\colon S \longrightarrow \mathbb{P}^1$ the morphism defined by Σ

$\beta\colon S^+ \longrightarrow \mathbb{P}^1$ the morphism defined by Σ^+

$p\colon X \longrightarrow S$ the first projection

$q\colon X \longrightarrow S^+$ the second projection

$\sigma := \alpha \circ p$

$\tau := \beta \circ q$

$\varphi := \varphi_{(S,\pi,\Sigma)}$

$\varphi^+ := \varphi_{(S^+,\lambda,\Sigma^+)}$

Remark that general fibres of α and β are isomorphic to $\mathbb{P}^1$. Let $F = \alpha^{-1}(x)$ (resp. $F^+ = \beta^{-1}(y)$) be a general member of Σ (resp. Σ^+). Then $\sigma^{-1}(x) \xrightarrow{p} \alpha^{-1}(x) \simeq \mathbb{P}^1$ is a nonsingular elliptic $K3$ surface, and similarly for $\tau^{-1}(y) \xrightarrow{q} \beta^{-1}(y) \simeq \mathbb{P}^1$. Let $\{P_1, P_2\} \in \mathbb{P}^1$ (resp. $\{Q_1, Q_2\} \in \mathbb{P}^1$) be the ramification values with respect to $\pi|_F\colon F \longrightarrow \mathbb{P}^1$ (resp. $\lambda|_{F^+}\colon F^+ \longrightarrow \mathbb{P}^1$).

Proposition(2.10) Assume that 1) $P_1 = Q_1$ and $P_2 = Q_2$, 2) (2.1) and (2.2) holds. Then $D := F \times_{\mathbb{P}^1} F^+ \subset S \times_{\mathbb{P}^1} S^+ = X$ consists of two irreducible components D_1, D_2 with the following properties:

a) D_1 and D_2 are nonsingular rational curves with $(D_1.D_2)_{\sigma^{-1}(x)} = 2$, $(D_1.D_2)_{\tau^{-1}(y)} = 2$. Moreover D_1 intersects with D_2 transversely.

b) $N_{D_i/X} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ or $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$.

Moreover if we assume that $\deg \varphi = 1$ or $\deg \varphi^+ = 1$, then D_i $(1 \leq i \leq 2)$ is an isolated rational curve.

Proof.

The proof of a) and b) is immediate from our assumption. Suppose that D_1 moves in X. If $N_{D_1/X} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, then D_1 is always rigid, hence we may assume that $N_{D_1/X} \simeq \mathcal{O} \oplus \mathcal{O}(-2)$. Since D_1 moves, the Hilbert scheme $Hilb_X$ is smooth at $[D_1]$. Let $u\colon \mathcal{H} \longrightarrow H$ be the irreducible component of $Hilb_X$ passing through $[D_1]$ and the universal family over this component. $\mathcal{H}$ is irreducible and there is a natural morphism from $\mathcal{H}$ to X. We can regard its image as a Weil divisor on X. Since X is factorial by Remark preceeding Definition (2.5), this is a Cartier divisor, which we denote by $\mathcal{D}$.

Then we have the following diagram:

$$\begin{array}{ccccc} \mathcal{D} & \subset & X & & \\ {\scriptstyle \sigma\times\tau|_{\mathcal{D}}}\downarrow & & \downarrow{\scriptstyle \sigma\times\tau} & & \\ G & \underset{i}{\subset} & \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow[\varphi \times \varphi^+]{} & \mathcal{G} \times \mathcal{G} \end{array}$$

, where G is the image of $\mathcal{D}$ under $\sigma \times \tau$ and φ (resp. φ^+) is the ramification map with respect to (S,π,Σ) (resp. (S^+,λ,Σ^+)) defined in §2. First, G is an irreducible curve because $\mathcal{D}$ is the image of $\mathcal{H}$ and D_1 is contained in a fibre of $\sigma \times \tau$. Next, for a general point $P \in G$, $(\sigma \times \tau)^{-1}(P)$ is a singular fibre of type I_2 (i.e. two nonsingular rational curves intersecting at two points transversely). This is shown as follows. A general fibre of σ (resp. τ) is an elliptic $K3$ surface. D_1 deforms to a section s of a general fibre of σ, and s is, at the same time, a section of a general fibre of τ . Assume that $s \in \sigma^{-1}(x')$: general fibre and $s \in \tau^{-1}(y')$: general fibre. Then $\alpha^{-1}(x') \xrightarrow{\pi} \mathbb{P}^1$ and $\beta^{-1}(y') \xrightarrow{\lambda} \mathbb{P}^1$ are double covers of $\mathbb{P}^1$, respectively. Of course, both $\alpha^{-1}(x')$ and $\beta^{-1}(y')$ are nonsingular rational curves by definition. Let $\{P_1', P_2'\} \in \mathbb{P}^1$ (resp. $\{Q_1', Q_2'\}$) be the ramification value with respect to π (resp. λ). If $\{P_1', P_2'\}$ does not coincides with $\{Q_1', Q_2'\}$, then $\alpha^{-1}(x') \times_{\mathbb{P}^1} \beta^{-1}(y') =: \mathrm{t}$ is an irreducibe curve on $\sigma^{-1}(x')$ and $p|_{\mathrm{t}}: \mathrm{t} \longrightarrow \alpha^{-1}(x') \subset S$ is a double cover. Since t contains a section s of $p|_{\sigma^{-1}(x')}: \sigma^{-1}(x') \longrightarrow \alpha^{-1}(x')$, this is a contradiction. Hence $\{P_1', P_2'\}$ coincides with $\{Q_1', Q_2'\}$. In this case, t is a union of two sections s and s'. s and s' intersect at two points which corresponds to $\{P_1' \times_{\mathbb{P}^1} Q_1'$, $P_2' \times_{\mathbb{P}^1} Q_2'\}$. Since $\mathrm{t}^2 = 0$, $s^2 = -2$ and $s'^2 = -2$ on the elliptic $K3$ surface $\sigma^{-1}(x')$, s and s' intersect at these points transversely. Therefore, for a general point $P \in G$, $(\sigma \times \tau)^{-1}(P)$ is a singular fibre of type I_2. Furthermore, from the above observation, we conclude that $(\varphi \times \varphi^+) \circ i$ factors through $\Delta \subset \mathscr{G} \times \mathscr{G}$.

Let us show that $\mathscr{D}$ is a pull-back of a divisor on $\mathbb{P}^1 \times \mathbb{P}^1$ by $\sigma \times \tau$. Let P be a general point of G and let us write $(\sigma \times \tau)^{-1}(P) = C_1 + C_2$. Then $C_1 + C_2$ is contained in a smooth fibre F of σ , and $\mathscr{D}|_F = \mathrm{a}C_1 + \mathrm{b}C_2 +$ (effective divisors on F disjoint from C_1 and C_2). Since (a general fibre of $\sigma \times \tau$. $\mathscr{D}$) = 0, we have $(C_1 + C_2 . \mathscr{D}) = 0$. Here we recall that X has two canonical projection $p: X \longrightarrow S$ and $q: X \longrightarrow S^+$. Then $p(C_i)$ (resp. $q(C_i)$) coincides with a fibre of $\alpha: S \longrightarrow \mathbb{P}^1$ (resp. $\beta : S^+ \longrightarrow \mathbb{P}^1$) for i = 1,2. On the other hand, by (2.4), $\mathcal{O}(\mathscr{D}) \simeq p^*L \otimes q^*M$ for an effective divisor L on S and an effective divisor M on S^+, which implies that $(\mathscr{D} . C_1) \geq 0$, $(\mathscr{D} . C_2) \geq 0$ and consequently that $(\mathscr{D} . C_1) = 0$ and $(\mathscr{D} . C_2) = 0$. Since C_i's are (-2)-curves on the surface F, we have $\mathscr{D}|_F = (\mathrm{a}C_1 + \mathrm{a}C_2 ; \mathrm{a} \geq 0)$

+ (effective divisors disjoint from C_i's). It is easy to see $\mathcal{O}(\mathcal{D})|_{C_1+C_2} \simeq \mathcal{O}_{C_1+C_2}$. Hence we have shown that except for finite fibres of $\sigma \times \tau$, the restriction of $\mathcal{O}(\mathcal{D})$ to fibres of $\sigma \times \tau$ are trivial. By the see-saw lemma, $\mathcal{D}$ is a pull-back of a divisor by $\sigma \times \tau$ in codimension one. Hence we have $\mathcal{O}(\mathcal{D}) \simeq (\sigma \times \tau)^*\mathcal{O}(G)$. Let us denote by p_1 (resp. p_2) the first projection (resp. the second projection) of $\mathbb{P}^1 \times \mathbb{P}^1$. Write $L = \mathcal{O}_S(F)$ and $M = \mathcal{O}_{S^+}(F^+)$. Then we have $(\sigma \times \tau)^* p_1^* \mathcal{O}(1) = p^*L$ and $(\sigma \times \tau)^* p_2^* \mathcal{O}(1) = q^*M$. Thus $\mathcal{O}(G) = p_1^* \mathcal{O}(\mathrm{m}) \otimes p_2^* \mathcal{O}(\mathrm{n})$ and $\mathcal{O}(\mathcal{D}) = p^*L^{\otimes \mathrm{m}} \otimes q^*M^{\otimes \mathrm{n}}$ for positive integers m and n. Since $(\varphi \times \varphi^+)\circ\iota$ factors through $\Delta \subset \mathcal{G} \times \mathcal{G}$ and $\deg \varphi$ or $\deg \varphi^+ = 1$, we have $\mathrm{m} = 1$ or $\mathrm{n} = 1$, and hence $G \simeq \mathbb{P}^1$.
We have the following commutative diagram:

$$\begin{array}{ccc}
\tilde{\mathcal{D}} & \xrightarrow[\nu]{\text{normalization}} & \mathcal{D} \\
\downarrow j & & \downarrow \sigma \times \tau|_{\mathcal{D}} \\
\tilde{G} & \xrightarrow{k} & G \simeq \mathbb{P}^1
\end{array}$$

, where $k\circ j$ is the Stein factorization of $\tilde{\mathcal{D}} \longrightarrow \mathbb{P}^1$. Since a general fibre of $\sigma \times \tau|_{\mathcal{D}}$ is of type I_2, a general fibre of j is isomorphic to $\mathbb{P}^1$. Hence we have $\deg k = 2$. Since every fibre of $\sigma \times \tau$ is reduced, every fibre of $\sigma \times \tau|_{\mathcal{D}}$ is also reduced. Let f be an arbitrary fibre of $\sigma \times \tau|_{\mathcal{D}}$. Then a general point $P \in$ f is a smooth point of f because f is reduced. Since $\sigma \times \tau|_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathbb{P}^1$ is a flat morphism, we conclude that $\mathcal{D}$ is smooth at P, which implies that the normalization of $\mathcal{D}$ makes effect only on finite points of every fibre. By Hurwitz formula, k is ramified over some points because $G \simeq \mathbb{P}^1$. Let $P \in \mathbb{P}^1$ be one of these points. Then $(k\circ j)^{-1}(P)$ is a multiple divisor on $\tilde{\mathcal{D}}$. On the other hand $(k\circ j)^{-1}(P) = \nu^{-1}\circ(\sigma \times \tau|_{\mathcal{D}})^{-1}(P)$ and the right side is not a multiple divisor by the above observation, which is a contradiction. Therefore, D_1 is isolated under the assumption of the proposition. Q.E D.

Definition(2.11) Let C be an nonsingular rational curve on a 3-fold X. Assume that X is smooth near C and that $N_{C/X} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Blowing up X along C, we can produce a $\mathbb{P}^1 \times \mathbb{P}^1$ as a exceptional divisor. Blowing down it to other directions, we have a threefold bimeromorphic to X. This process is called a *flop*. For the definition of a flop in more general situations, see [4],[8].

Definition(2.12) The Weierstrass model $W(K_T,a,b)$ over a surface T is the divisor on $\mathbb{P}_T(\mathcal{O}_T\oplus K_T^{\otimes 2}\oplus K_T^{\otimes 3})$ defined by the equation $Y^2Z - (X^3 + aXZ^2 + bZ^3)$, where X,Y and Z are natural injections

$$\begin{aligned} K_T^{\otimes 2} &\longrightarrow \mathcal{O}_T\oplus K_T^{\otimes 2}\oplus K_T^{\otimes 3} \\ K_T^{\otimes 3} &\longrightarrow \mathcal{O}_T\oplus K_T^{\otimes 2}\oplus K_T^{\otimes 3} \\ \mathcal{O}_T &\longrightarrow \mathcal{O}_T\oplus K_T^{\otimes 2}\oplus K_T^{\otimes 3} \end{aligned}$$

, respectively, and where a and b are sections of $K_T^{\otimes -4}$ and $K_T^{\otimes -6}$, respectively. Moreover we assume that $4a^3 + 27b^2$ is not identically zero. We shall denote $W(K_T, a, b)$ simply by W.

§ 3. Rational curves

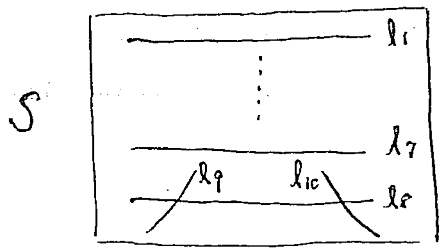
1. In this paragraph, we shall define the birational map ϕ between $X = S \times_{\mathbb{P}^1} S^+$ and a singular Weierstrass model μ: $W \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

Let π: $S \longrightarrow \mathbb{P}^1$ be a rational elliptic surface with sections. We assume that all singular fibres are irreducible. Then S has the representation as a nine points blow-up of $\mathbb{P}^2$. Thus there are (-1)-curves ℓ_i $(1 \le i \le 10)$ with the following properties:

(1) ℓ_is $(i \neq 8)$ are exceptional curves of the nine points blow-up of $\mathbb{P}^2$.

(2) $(\ell_8 . \ell_9) = (\ell_{10} . \ell_8) = 1$. $(\ell_i . \ell_8) = 0$ for other i.

The blow-down of $\ell_1, \ldots, \ell_8$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.



Let λ : $S^+ \longrightarrow \mathbb{P}^1$ be a rational elliptic surface with sections. We assume that all singular fibres are irreducible. Pick up one section $m_1 \subset S^+$ of λ (which is a (-1)-curve on S^+.). The birational map ϕ_{ℓ, m_1}(which depends on $\ell = (\ell_i)$ and m_1) is defined as follows.

Let γ be the flop of the curves $\ell_i \times_{\mathbb{P}^1} m_1$ $(1 \le i \le 8)$. Then we have the projective threefold X^+ birational to $X = S \times_{\mathbb{P}^1} S^+$. Let E_i $(1 \le i \le 8)$ be the strict transform of $\ell_i \times_{\mathbb{P}^1} S^+$ $(1 \le i \le 8)$ by γ. These E_is can be contracted to eight points and we obtain a singular Weierstrass model μ: $W \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. We denote by $\phi_{(\ell, m_1)}$ this birational map between X and W. The strict transform of $S \times_{\mathbb{P}^1} m_1$ by $\phi_{(\ell, m_1)}$ is the canonical section of μ.

2. We shall explain the basic situation. Let $\pi: S \longrightarrow \mathbb{P}^1$ and $\lambda: S^+ \longrightarrow \mathbb{P}^1$ be rational elliptic surfaces with sections. We assume that every singular fibre of π and λ is irreducible. We denote by ℓ and ℓ' (resp. m and m') two (-1)-curves on S (resp. S^+) for which $(\ell. \ell') = 1$ (resp. $(m, m') = 1$). Assume that

(3.1) generic fibre of π and λ are not isogenous, and

(3.2) $\pi(P) = \lambda(Q)$, where $P = \ell \cap \ell'$ and $Q = m \cap m'$.

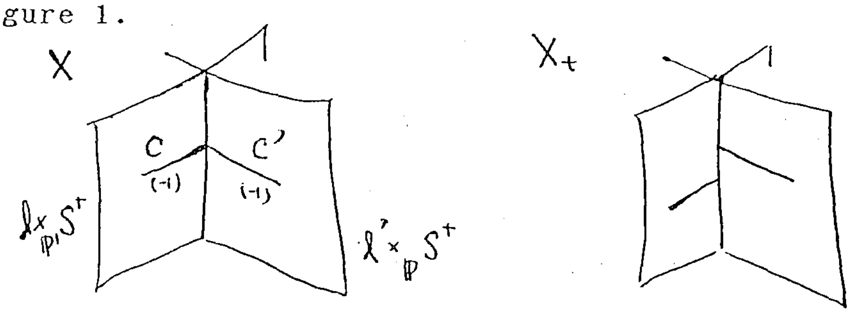
Let Δ be a small disk around $0 \in \mathbb{C}$. Let $f: \mathfrak{S} \longrightarrow \Delta \times \mathbb{P}^1$ and $g: \mathfrak{S}^+ \longrightarrow \Delta \times \mathbb{P}^1$ be the deformation of $\pi: S \longrightarrow \mathbb{P}^1$ and $\lambda: S^+ \longrightarrow \mathbb{P}^1$, respectively.

$$\begin{array}{ccc} \mathfrak{S} & \longleftarrow & \mathfrak{S}_0 = S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_0} \\ \Delta \times \mathbb{P}^1 & \leftarrow & \{0\} \times \mathbb{P}^1 \end{array} \qquad \begin{array}{ccc} \mathfrak{S}^+ & \longleftarrow & \mathfrak{S}_0^+ = S^+ \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_0} \\ \Delta \times \mathbb{P}^1 & \leftarrow & \{0\} \times \mathbb{P}^1 \end{array}$$

Since ℓ and ℓ' (resp. m and m') are stable submanifolds in S (resp. S^+), they are deformed to (-1)-curves ℓ_t and ℓ'_t on $\mathfrak{S}_t$ (resp. m_t and m'_t on $\mathfrak{S}_t^+$); $t \in \Delta - \{0\}$. Here we impose the following general condition:

(3.3) $f_t(P_t) \neq g_t(Q_t)$ for $t \in \Delta - \{0\}$, where $P_t := \ell_t \cap \ell'_t$ and $Q_t := m_t \cap m'_t$.

Figure 1.



Let us employ the following notation:

$C := \ell \times_{\mathbb{P}^1} m \subset X = S \times_{\mathbb{P}^1} S^+$

$C' := \ell' \times_{\mathbb{P}^1} m' \subset X$

$C \cup C'$:= the reduced subscheme of X whose support is $C \cup C'$

For $t \in \Delta - \{0\}$,

$X_t := \mathfrak{S}_t \times_{(\{t\} \times \mathbb{P}^1)} \mathfrak{S}_t^+$

$X_t^+ :=$ the threefold obtained from X_t by the flop γ_t of $\ell_t' \times_{\mathbb{P}^1} m_t'$ on X_t

$X^+ :=$ the threefold obtained from X by the flop γ of $C' \subset X$

$D :=$ the strict transform of C by γ

$E :=$ the strict transform of $\ell \times_{\mathbb{P}^1} S^+$ by γ

For $t \in \Delta$,

$\alpha_t : \mathfrak{S}_t \longrightarrow \mathbb{P}^1$, the morphism defined by the linear system $|\ell_t + \ell_t'|$

$\beta_t : \mathfrak{S}_t^+ \longrightarrow \mathbb{P}^1$, the morphism defined by the linear system $|m_t + m_t'|$

$p_t : X_t \longrightarrow \mathfrak{S}_t$, the first projection

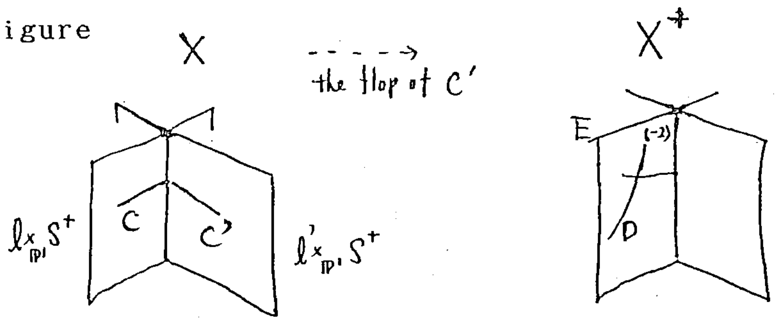
$q_t : X_t \longrightarrow \mathfrak{S}_t^+$, the second projection

$\sigma_t := \alpha_t \circ p_t$

$\tau_t := \beta_t \circ q_t$

Remark. General fibres of α_t and β_t are isomorphic to $\mathbb{P}^1$, and general fibres of σ_t and τ_t are elliptic K-3 surfaces.

Figure



Lemma(3.4) *The following implication holds.*

a) $C \cup C'$ *is isolated in* X.

$\longrightarrow$b) $D \subset X^+$ *is an isolated* (-2)*-curve* (i.e. $N_{D/X^+} \simeq \mathcal{O}(-2) \oplus \mathcal{O}$ or $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$)

$\longrightarrow$c) *D can be deformed to an isolated (-2)-curve D_t on X_t^+ ($t \in \Delta - \{0\}$)*

Furthermore γ_t^{-1} is an isomorphism in a neighbourhood of D_t.

Proof.

a) $\longrightarrow$ b) Suppose that $D \subset X^+$ moves. If $N_{D/X^+} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, then D is always isolated, which implies that $N_{D/X^+} \simeq \mathcal{O}(-2) \oplus \mathcal{O}$. Let H be the germ of $Hilb_{X^+/\mathbb{C}}$ at $[D]$. Then , since D moves in X^+, $\dim H = 1$ and H is isomorphic to $(\mathbb{C}^1, 0)$. Denoting by $\mathcal{H}$ the universal family on H, we obtain the following diagram:

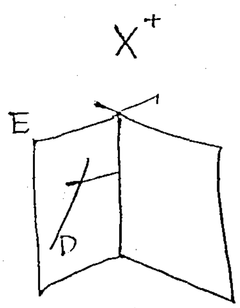
$$\begin{array}{ccccc} \mathcal{H} & \longrightarrow & X^+ \times H & \xrightarrow{p_1} & X^+ \\ & & {\scriptstyle p_2}\downarrow & & \\ & & H & & \end{array}$$

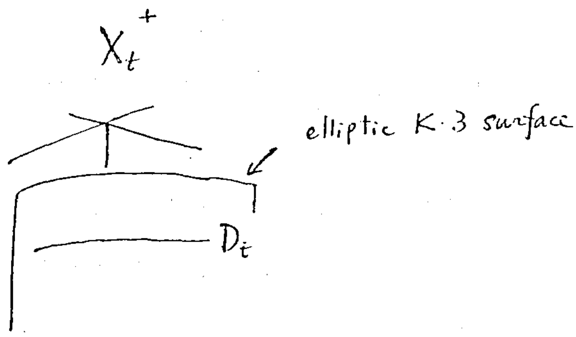
In the above diagram $p_1|_{\mathcal{H}} : \mathcal{H} \longrightarrow X^+$ is unramified at each point on $p_2^{-1}([D])$ because $N_{D/X^+} \simeq \mathcal{O}(-2) \oplus \mathcal{O}$. Hence $p_1(\mathcal{H})$ is smooth in a neighbourhood of D. Since $h^0(N_{D/E}) = 0$, $p_1(\mathcal{H})$ intersects with E transversely, and $p_1(\mathcal{H})|_E = D$. Indeed if $p_1(\mathcal{H})|_E = \mathrm{n}D$ for $\mathrm{n} \geq 2$, then we have a non-trivial extension of D to the second order. This contradicts the fact that $h^0(N_{D/E}) = 0$. Moreover $p_1(\mathcal{H})$ has the D as a ruling. Let $\mathcal{B}$ be the strict transform of $p_1(\mathcal{H})$ by γ^{-1}. Then $\mathcal{B}$ contains C and C' as (-1)-curves, which implies that $C \cup C'$ moves in X. This contradicts the condition a).

b) $\longrightarrow$ c) The first part of c) follows from §2 of [2]. The latter part of c) follows from the assumption (3.3). Q.E.D.

Remark. In c), D_t can be regarded as an isolated (-2)-curve on X_t because γ^{-1} is an isomorphism near D_t. Hence by Lemma(3.4) if a) is satisfied, then we can find an isolated (-2)-curve on X_t.

Figure





We shall denote by Σ (resp. Σ^+) the linear system on S (resp. S^+) defined by $\ell + \ell'$(resp. $m + m'$). Then the argument in (2.5) is applied to the triplet (S,π,Σ) and (S^+,λ,Σ^+).

Proposition(3.5) *Let S and S^+ be as above. Suppose that the triplet (S,π,Σ) is generally chosen such that Lemma(2.7) holds. Then $C \cup C'$ is isolated in X.*

Proof.

Suppose that $C \cup C'$ moves in X. Let us denote by N, the normal bundle $N_{C \cup C'/X}$ and write $P = C \cap C'$, $j: C \longrightarrow C \cup C'$ and $j': C' \longrightarrow C \cup C'$. Then we have

$$0 \longrightarrow N \longrightarrow j_*(N|_C) \oplus j'_*(N|_{C'}) \xrightarrow{\Psi} \mathbb{C}(P)^{\oplus 2} \longrightarrow 0$$

, where $N|_C \simeq \mathcal{O}(-1) \oplus \mathcal{O}$ and $N|_{C'} \simeq \mathcal{O}(-1) \oplus \mathcal{O}$. Since $\Phi : j_*(N|_C) \otimes \mathbb{C}(P) \longrightarrow j_*(N|_C) \otimes \mathbb{C}(P) \oplus j'_*(N|_{C'}) \otimes \mathbb{C}(P) \xrightarrow{\Psi} \mathbb{C}(P)^{\oplus 2}$ and Φ': $j'_*(N|_{C'}) \otimes \mathbb{C}(P) \longrightarrow \mathbb{C}(P)^{\oplus 2}$ are both isomorphisms, we have $h^0(N) \leq 1$. Since $C \cup C'$ moves in X, $Hilb_{X/\mathbb{C}}$ is smooth of dim 1 at $[C \cup C']$. We note here that X has two fibrations σ_0 and τ_0 whose general fibres are elliptic K3 surfaces.(See the notation above.) $C \cup C'$ is contained in sigular fibres of σ_0 and τ_0, respectively.

Since $C \cup C'$ does not move in these singular fibre, $C \cup C'$is deformed to a section of a general fibre(an elliptic K3 surface) of σ_0(resp. τ_0). Let us denote by α: $\mathcal{H} \longrightarrow H$, the irreducible component of $Hilb_{X/\mathbb{C}}$ passing through $[C \cup C']$ and the universal family over this component. Then a general fibre of α is isomorphic to $\mathbb{P}^1$, which implies that $\mathcal{H}$ is irreducible. There is a natural morphism from $\mathcal{H}$ to X, and we can regard its image as a Weil divisor on X. Since X is factorial by Remark in §2, this is a Cartier divisor, which we denote by $\mathcal{D}$. Then we have the following diagram:

$$\begin{array}{ccccc} \mathcal{D} & \subset & X & & \\ \sigma_0\times\tau_0|_{\mathcal{D}} \downarrow & & \downarrow \sigma_0\times\tau_0 & & \\ G & \underset{i}{\subset} & \mathbb{P}^1\times\mathbb{P}^1 & \xrightarrow[\varphi\times\varphi^+]{} & \mathcal{G}\times\mathcal{G} \end{array}$$

, where G is the image of $\mathcal{D}$ under $\sigma_0\times\tau_0$ and φ (resp. φ^+) is the ramification map with respect to (S,π,Σ) (resp. (S^+,λ,Σ^+)) defined in §2. First, G is an irreducible curve because $\mathcal{D}$ is the image of $\mathcal{H}$ and $C\cup C'$ is contained in a fibre of $\sigma_0\times\tau_0$. Next, for a general point $P\in G$, $(\sigma_0\times\tau_0)^{-1}(P)$ is a singular fibre of type I_2 (i.e. two nonsingular rational curves intersecting at two points transversely). This is shown as follows. A general fibre of σ_0 (resp. τ_0) is an elliptic $K3$ surface. $C\cup C'$ deforms to a section s of a general fibre of σ_0, and s is, at the same time, a section of a general fibre of τ_0. Hence we have the situation in the proof of Proposition(2.10), and consequently for a general point $P\in G$, $(\sigma_0\times\tau_0)^{-1}(P)$ is a singular fibre of type I_2. From this, we can use the same argument of the proof of Proposition(2.10), which deduces the contradiction. Therefore, $C\cup C'$ is isolated in X. Q.E.D.

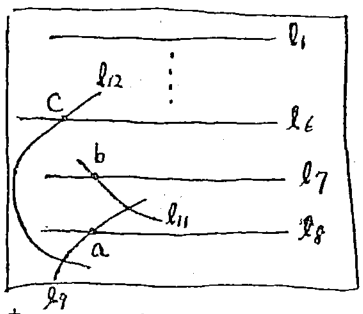
3. In the following we shall find four mutually disjoint, isolated (-2)-curves C_i $(1\le i\le 4)$ on a singular Weierstrass model $\mu\colon W\longrightarrow\mathbb{P}^1\times\mathbb{P}^1$. These curves will have the following intersection numbers:

$$(3.6)\quad \begin{array}{lll} (\Sigma\ ,\ C_1) = 0 & (\mu^* p_1^* \mathcal{O}(1),\ C_1) = 1 & (\mu^* p_2^* \mathcal{O}(1),\ C_1) = 0 \\ (\Sigma\ ,\ C_2) = 0 & (\mu^* p_1^* \mathcal{O}(1),\ C_2) = 0 & (\mu^* p_2^* \mathcal{O}(1),\ C_2) = 1 \\ (\Sigma\ ,\ C_3) = 1 & (\mu^* p_1^* \mathcal{O}(1),\ C_3) = 0 & (\mu^* p_2^* \mathcal{O}(1),\ C_3) = 1 \\ (\Sigma\ ,\ C_4) = 1 & (\mu^* p_1^* \mathcal{O}(1),\ C_4) = 1 & (\mu^* p_2^* \mathcal{O}(1),\ C_4) = 0\ , \end{array}$$

where p_1(resp. p_2) is the first(resp. second) projection of $\mathbb{P}^1 \times \mathbb{P}^1$ and Σ is the canonical section of Σ.

Let π: $S \longrightarrow \mathbb{P}^1$ be a rational elliptic surface with sections whose singular fibres are irreducible. S is a nine points blow-up of $\mathbb{P}^2$ and a eight points blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$. Let ℓ_i $(1 \le i \le 10)$ be the same as in **1.** of this section, and let ℓ_i $(i = 11, 12)$ be the (-1)-curves on S for which $(\ell_{11}.\ell_9) = (\ell_{12}.\ell_9) = (\ell_{11}.\ell_7) = (\ell_{12}.\ell_6) = 1$, $(\ell_{11}.\ell_j) = 0$ $(j \ne 7,9)$ and $(\ell_{12}.\ell_j) = 0$ $(j \ne 6,9)$. We write $a = \ell_8 \cap \ell_9$, $b = \ell_7 \cap \ell_{11}$ and $c = \ell_6 \cap \ell_{12}$. It is easy to see the existence of these curves.

Figure



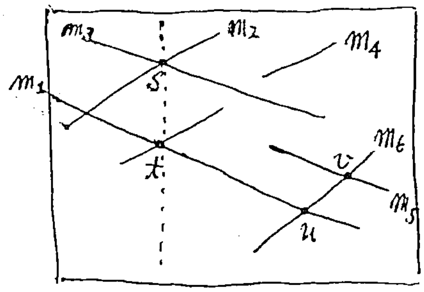
Let λ: $S^+ \longrightarrow \mathbb{P}^1$ be a rational surface with sections whose singular curves are irreducible. Suppose that m_i $(1 \le i \le 6)$ be the (-1)-curves on S^+ for which

$$(3.7)\quad \begin{array}{l} (m_i\,.\,m_j) = 0 \text{ if } (i,j) = (\text{odd, odd}) \text{ or } (\text{even, even}) \text{ with } i \ne j \\ (m_1\,.\,m_2) = (m_1\,.\,m_4) = (m_1\,.\,m_6) = 1 \\ (m_3\,.\,m_2) = 1 \qquad (m_3\,.\,m_4) = (m_3\,.\,m_6) = 0 \\ (m_5\,.\,m_2) = (m_5\,.\,m_4) = 0 \qquad (m_5\,.\,m_6) = 1. \end{array}$$

Writing $s = m_2 \cap m_3$, $t = m_1 \cap m_4$, $u = m_1 \cap m_6$ and $v = m_5 \cap m_6$ we suppose one more condition:

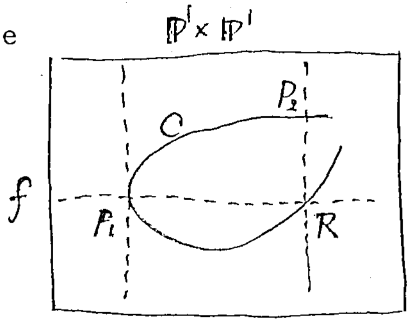
(3.8) $\lambda(s) = \lambda(t)$.

Figure



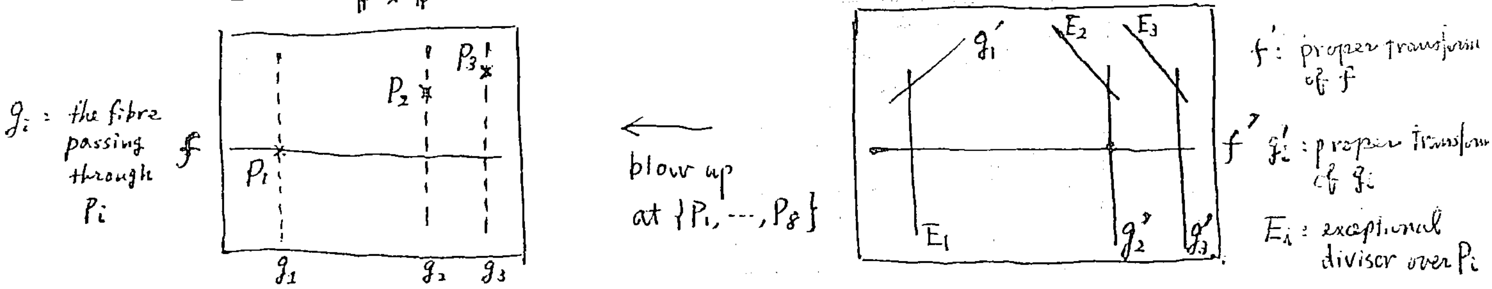
In order to construct ths S^+ satisfying (3.6) and (3.7), we must choose suitable eight points on $\mathbb{P}^1 \times \mathbb{P}^1$ (which are, of course, the base points of certain linear pencil on $\mathbb{P}^1 \times \mathbb{P}^1$) and blow up these points. Let us look at the following figure.

Figure



Here C is a nonsingular elliptic curve which is a divisor of (2,2) type in $\mathbb{P}^1 \times \mathbb{P}^1$. Let us denote by p_1(resp. p_2) the first (resp. second) projection of $\mathbb{P}^1 \times \mathbb{P}^1$. C is a double cover of $\mathbb{P}^1$ with respect to p_1 and p_2. Let P_1 be a ramification point of the double cover $p_1|_C$. Let f be the fibre of p_2 passing through P_1 and R a intersection of f and C. If we choose C generally, then C intersects with the fibre g of p_1 passing through R, transversely. Write $C \cap g = \{R, P_2\}$. Take five points $\{P_3, \ldots, P_7\}$ on C disjoint from $\{P_1, P_2, R\}$. Then the divisor of (2,2) type passing through $\{P_1, \ldots P_7\}$ forms a linear pencil Σ. Let $\{C, C'\}$ be a basis of Σ. Then $C \cap C' = \{P_1, \ldots, P_7, P_8\}$, where P_8 coincides with none of $\{P_1, \ldots, P_7\}$. Blowing up $\{P_1, \ldots, P_8\}$, we have an elliptic fibration λ with sections. If we impose certain general conditions, then the elliptic fibration obtained has only irreducible fibres.

Figure



Then we may set $m_1 = f'$, $m_2 = E_1$, $m_3 = g_1'$, $m_4 = g_2'$, $m_5 = g_3'$ and $m_6 = E_3$. We assume the following condition (3.1),(3.2*).

(3.1) Generic fibres of π and λ are not isogenous.

(3.2*) $\pi(a) = \lambda(s)(= \lambda(t))$, $\pi(b) = \lambda(u)$ and $\pi(c) = \lambda(v)$.

Remark. By replacing π by $\sigma\circ\pi$ for a suitable $\sigma \in Aut(\mathbb{P}^1)$, (3.2*) is always satisfied.

Let f: $\mathfrak{S} \longrightarrow \Delta \times \mathbb{P}^1$ and g: $\mathfrak{S}^+ \longrightarrow \Delta \times \mathbb{P}^1$ be the deformations of π: $S \longrightarrow \mathbb{P}^1$ and λ: $S^+ \longrightarrow \mathbb{P}^1$, respectively, for which (3.3) holds. Then Lemma(3.4) and Proposition(3.5) is applied for each

$$D_1 = (9,\ 2) \cup (8,\ 3)$$
$$D_2 = (9,\ 4) \cup (8,\ 1)$$
$$D_3 = (11,\ 6) \cup (7,\ 1)$$
$$D_4 = (12,\ 6) \cup (6,\ 5),$$

where (i,j) means $\ell_i \times_{\mathbb{P}^1} m_j$. Since $D_i (1 \le i \le 4)$ are disjoint from each other, we have mutually disjoint, isolated (-2)-curves $D_{i,t} (1 \le i \le 4)$ on X_t ($t \ne 0$). As is explained in (§3, 1), there is a birational map $\phi_{(\ell_t,\ m_{1t})}$ between X_t and a singular Weierstrass model W_t over $\mathbb{P}^1 \times \mathbb{P}^1$. But in a neighbourhood of each $D_{i,t} (1 \le i \le 4)$ $\phi_{(\ell_t,\ m_{1t})}$ is an isomorphism. Therefore we can find mutually disjoint (-2)-curves on W; $C_i = \phi_{(\ell_t,\ m_{1t})}(D_{i,t})$ $(1 \le i \le 4)$. By the construction we have (3.6). W can be deformed to a nonsingular

Weierstrass model over $\mathbb{P}^1 \times \mathbb{P}^1$, which we call W by abuse of language. Then the four curves constructed above is also deformed to isolated (-2)-curves on W (one curve may split up into disjoint several curves.). Hence we can find mutually disjoint, isolated (-2)-curves C_i $(1 \leq i \leq 4)$ on a nonsingular Weierstrass model W , and these curves satisfy (3.6).

§4. Deformation of rational curves

Let W be a nonsingular Weierstrass model over $\mathbb{P}^1 \times \mathbb{P}^1$. Let C denote one of the isolated (-2)-curves constructed above. If $N_{C/W} \simeq \mathcal{O}(-2) \oplus \mathcal{O}$, then we must deform (C, W) so that C splits into a disjoint union of (-1, -1)-curves to use Friedman's argument. In this section we observe the deformation of (C, W).

Lemma(4.1). *Let W be a nonsingular Weierstrass model over $\mathbb{P}^1\times \mathbb{P}^1$. Then W is unobstructed, that is, the moduli number* $\mathrm{m}(W)$ *of W can be defined and* $\mathrm{m}(W) = h^1(W, \Theta)$.

Proof.

By the definition of a Weierstrass model, W is embedded in $\mathbb{P} = \mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O} \oplus K_{\mathbb{P}^1\times\mathbb{P}^1}^{\otimes 2} \oplus K_{\mathbb{P}^1\times\mathbb{P}^1}^{\otimes 3})$. We shall denote $K_{\mathbb{P}^1\times\mathbb{P}^1}$ simply by K. Let p denote the projection of $\mathbb{P}$ to $\mathbb{P}^1\times \mathbb{P}^1$. Taking cohomologies of the exact sequence

$$0 \longrightarrow \Theta_W \longrightarrow \Theta_{\mathbb{P}}|_W \longrightarrow \mathcal{O}_W(W) \longrightarrow 0,$$

we have the exact sequence

$$0 \longrightarrow H^0(W, \Theta_{\mathbb{P}}|_W) \xrightarrow{\psi} H^0(W, \mathcal{O}_W(W)) \xrightarrow{\delta} H^1(W, \Theta_W).$$

The injectivity of ψ follows from the fact that $h^0(W, \Theta_W) = h^0(W, \Omega^2_W) = h^2(W, \mathcal{O}_W) = 0$. First we shall show that δ is surjective. Since the Euler number $e(W)$ of W is equal to $-60c_1^2(\mathbb{P}^1\times \mathbb{P}^1) = -480$ by [6, Th(3.10)] and $h^{1,1}(W) = 3$, we obtain $h^1(W, \Theta_W) = 243$. Since $\mathcal{O}_{\mathbb{P}}(W) = \mathcal{O}_{\mathbb{P}}(3)\otimes p*(-6K)$, a direct calculation using the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\mathbb{P}}(W) \longrightarrow \mathcal{O}_W(W) \longrightarrow 0$$

shows that $h^0(W, \mathcal{O}_W(W)) = 334$. Consider the following exact sequences

$$(4.2) \quad 0 \longrightarrow \Theta_{\mathbb{P}/\mathbb{P}^1\times \mathbb{P}^1}|_W \longrightarrow \Theta_{\mathbb{P}}|_W \longrightarrow p*\Theta_{\mathbb{P}^1\times \mathbb{P}^1}|_W \longrightarrow 0$$

$$(4.3) \quad 0 \longrightarrow \mathcal{O}_W \longrightarrow p*\varepsilon^{-1}\otimes \mathcal{O}_W(1) \longrightarrow \Theta_{\mathbb{P}/\mathbb{P}^1\times \mathbb{P}^1}|_W \longrightarrow 0$$

, where $\varepsilon = \mathcal{O} \oplus K^{\otimes 2}\oplus K^{\otimes 3}$. By (4.2) we have $h^0(\Theta_{\mathbb{P}}|_W) \leq h^0(\Theta_{\mathbb{P}/\mathbb{P}^1\times W}) + h^0(p*\Theta_{\mathbb{P}^1\times \mathbb{P}^1}|_W)$. On the other hand, $h^0(\Theta_{\mathbb{P}/\mathbb{P}^1\times \mathbb{P}^1}|_W) = h^0(p*\varepsilon^{-1}\otimes \mathcal{O}_W(1)) - 1$ by (4.3). An easy computation shows that $h^0(\Theta_{\mathbb{P}}|_W) \leq 91$. As a consequence, δ is surjective.

Next we shall show that $h^1(\mathcal{O}_W(W)) = 0$. Noting that $\mathcal{O}_W(W) = \mathcal{O}_W(W) = \mathcal{O}_W(3)\otimes p*(-6K)$ and applying Leray spectral sequence, we have

$$0 \longrightarrow H^1(p_*\mathcal{O}_W(3)\otimes \mathcal{O}(-6K)) \longrightarrow H^1(\mathcal{O}_W(W))$$

$$\longrightarrow H^0(R^1p_*\mathcal{O}_W(3)\otimes \mathcal{O}(-6K)).$$

But the first term and the last term of the sequence are shown to be zero. Thus we conclude that $H^1(\mathcal{O}_W(W)) = 0$. Since $H^1(\mathcal{O}_W(W)) = H^1(N_{W/\mathbb{P}}) = 0$, the Hilbert scheme $Hilb_{\mathbb{P}/k}$ is smooth at $[W]$, where $[W]$ is the point on $Hilb_{\mathbb{P}/k}$ which corresponds to W. This implies that the universal family over $Hilb_{\mathbb{P}/k}$ is complete near $[W]$ by [5] because δ is surjective. Therefore, we conclude that $m(W) = h^1(W, \Theta_W)$. Q.E.D.

Let μ: $W \longrightarrow \mathbb{P}^1\times \mathbb{P}^1$ be a general nonsingular Weierstrass model over $\mathbb{P}^1 \times \mathbb{P}^1$, and let C be the same as above. Then $\mu(C)$ is a ruling f of $\mathbb{P}^1 \times \mathbb{P}^1$. If we write $H = \mu^{-1}(f)$, then $C \subset H$. In case $N_{C/W} \simeq \mathcal{O}(-2) \oplus \mathcal{O}$, we shall use

Proposition(4.4). *Let* $\mu \colon W \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ *as above, and* Σ *a canonical section of* μ. *Suppose that*

1) *C is a contractible rational curve with $N_{C/W} = \mathcal{O}(-2) \oplus \mathcal{O}$,*
2) *$C \subset H = \mu^{-1}(f)$, where f is a ruling of $\mathbb{P}^1 \times \mathbb{P}^1$, and C is a section of $\mu|_H \longrightarrow f$.*
3) *$(C, \Sigma) = 0$ or 1.*

Then W can be deformed so that C splits up into a disjoint union of rational curves with normal bundles $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

Proof.

According to [2,§4(c) Cor.(4.11)], it suffices to show that the natural map $H^1(\Theta_W) \longrightarrow H^1(\Theta_{W|C})$ is surjective. If this is shown, then we can apply the same argument of [2,§4(c)(4.11)] with the aid of Lemma(4.1) to prove the existence of a desired deformation of W. With the same notation of Lemma(4.1), we have

$$0 \longrightarrow \Theta_W \longrightarrow \Theta_{\mathbb{P}}|_W \longrightarrow \mathcal{O}_W(W) \longrightarrow 0.$$

Let I_C denote the defining ideal of C in W. Then from the above sequence we obtain

$$(4.5) \qquad \begin{array}{ccccc} & & & & H^1(\Theta_{\mathbb{P}}|_C) \\ & & & & \downarrow \\ & & H^2(\Theta_W \otimes I_C) & \xrightarrow{\beta} & H^2(\Theta_{\mathbb{P}}|_W \otimes I_C) \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow \\ H^1(\mathcal{O}_W(W)) & \longrightarrow & H^2(\Theta_W) & \longrightarrow & H^2(\Theta_{\mathbb{P}}|_W) \\ & & & & \downarrow \\ & & & & H^2(\Theta_{\mathbb{P}}|_C) = 0 \end{array}$$

We have shown in the proof of Lemma(4.1) that $H^1(\mathcal{O}_W(W)) = 0$. Since

$\dim C = 1$, $H^2(\Theta_{\mathbb{P}}|_C) = 0$. We shall show that $H^1(\Theta_{\mathbb{P}}|_C) = 0$. By Serre duality, $h^1(\Theta_{\mathbb{P}}|_C) = h^0(\Omega^1_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}^1}(-2))$, where we identify C with $\mathbb{P}^1$. Consider the following exact sequences:

$$0 \longrightarrow p^*\Omega^1_{\mathbb{P}^1\times\mathbb{P}^1} \longrightarrow \Omega^1_{\mathbb{P}} \longrightarrow \Omega^1_{\mathbb{P}/\mathbb{P}^1\times\mathbb{P}^1} \longrightarrow 0$$

$$0 \longrightarrow \Omega^1_{\mathbb{P}/\mathbb{P}^1\times\mathbb{P}^1} \longrightarrow p^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow 0,$$

where $\mathcal{E} = \mathcal{O} \oplus K^{\otimes 2} \oplus K^{\otimes 3}$ and p is the projection of $\mathbb{P}$ to $\mathbb{P}^1\times \mathbb{P}^1$. From the first sequence, we obtain

$$0 \longrightarrow H^0(p^*\Omega^1_{\mathbb{P}^1\times\mathbb{P}^1}\otimes \mathcal{O}_{\mathbb{P}^1}(-2)) \longrightarrow H^0(\Omega^1_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}^1}(-2))$$
$$\longrightarrow H^0(\Omega^1_{\mathbb{P}/\mathbb{P}^1\times\mathbb{P}^1}\otimes \mathcal{O}_{\mathbb{P}^1}(-2)).$$

Since $H^0(p^*\Omega_{\mathbb{P}^1\times\mathbb{P}^1}\otimes \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$, in order to show that $H^0(\Omega^1_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$ it suffices to prove that $H^0(\Omega^1_{\mathbb{P}/\mathbb{P}^1\times\mathbb{P}^1}\otimes \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$. But by the second sequence, it is enough to show that $h^0(p^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$. If we write $m = (C,\Sigma) \geq 0$, then it follows from the assumption(2) that

$$p^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(-2) = \mathcal{O}_{\mathbb{P}^1}(-3m-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-3m-6) \oplus \mathcal{O}_{\mathbb{P}^1}(-3m-8).$$

Thus $h^0(p^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$ follows, and consequently $h^1(\Theta_{\mathbb{P}}|_C) = 0$ is obtained. To prove the surjectivity of the map $H^1(\Theta_W) \longrightarrow H^1(\Theta_W|_C)$, we must show that α in (4.5) is injective. But since $h^1(\Theta_{\mathbb{P}}|_C) = 0$, it is equivalent to show the injectivity of β. Consider the following exact sequences:

(4.6)

$$\begin{array}{ccccccc} & & H^0(\mathcal{O}_W(W)) & & & & \\ & & \downarrow & & & & \\ & & H^0(\mathcal{O}_C(W)) & & & & \\ & & \downarrow & & & & \\ H^1(\Theta_{\mathbb{P}}|_W\otimes I_C) & \xrightarrow{\gamma} & H^1(\mathcal{O}_W(W)\otimes I_C) & \longrightarrow & H^2(\Theta_W\otimes I_C) & \xrightarrow{\beta} & H^2(\Theta_{\mathbb{P}}|_W\otimes I_C) \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

We may prove that γ is surjective. Let us write

$L = H^1(\mathcal{O}_W(W)\otimes I_C) = H^0(\mathcal{O}_C(W))/H^0(\mathcal{O}_W(W))$. Then we have the following commutative diagram:

$$\begin{array}{ccc} H^0(p_*\varepsilon^{-1} \otimes \mathcal{O}_C(1)) & \xrightarrow{\ \tau\ } & H^1(\Theta_{\mathbb{P}}|_W \otimes I_C) \\ \downarrow J & & \downarrow \\ H^0(\mathcal{O}_C(W)) & \longrightarrow & L \end{array}$$

Here $\mathcal{O}_C(1)$ is the restriction of $\mathcal{O}_{\mathbb{P}}(1)$ to C, and τ is the composition of the three maps: $H^0(p_*\varepsilon^{-1}\otimes \mathcal{O}_C(1)) \longrightarrow H^0(\Theta_{\mathbb{P}/\mathbb{P}^1\times\mathbb{P}^1}|_C)$, $H^0(\Theta_{\mathbb{P}/\mathbb{P}^1\times\mathbb{P}^1}|_C) \longrightarrow H^0(\Theta_{\mathbb{P}}|_C)$ and $H^0(\Theta_{\mathbb{P}}|_C) \longrightarrow H^1(\Theta_{\mathbb{P}}|_W\otimes I_C)$.(See (4.2) and (4.3).) J is the so called Jacobian map, and it is defined as follows:

Let $F = Y^2Z - (X^3 + aXZ^2 + bZ^3)$ be the defining equation of W in $\mathbb{P}$.(See Definition.) We write

$$H^0(p_*\varepsilon^{-1}\otimes \mathcal{O}_{\mathbb{P}}(1)) = H^0(\mathcal{O}_C(1))\oplus H^0(\mathcal{O}_C(1)\otimes p^*K^{-2})\oplus H^0(\mathcal{O}_C(1)\otimes p^*K^{-3}).$$

Then J is given by

$$(\ell_1, \ell_2, \ell_3) \longrightarrow \underset{H^0(\mathcal{O}_C(W)),}{\underset{\Cap}{\ell_1(\partial F/\partial Z) + \ell_2(\partial F/\partial X)}} + \ell_3(\partial F/\partial Y)$$

where

$\ell_1 \in H^0(\mathcal{O}_C(1))$,

$\ell_2 \in H^0(\mathcal{O}_C(1)\otimes p^*K^{-2})$,

$\ell_3 \in H^0(\mathcal{O}_C(1)\otimes p^*K^{-3})$.

First it follows that $L = 0$ if $(C, \Sigma) = 0$. For, we have $\mathcal{O}_W(W) \cong \mathcal{O}_W(3) \otimes \mu^*(-6K) \cong \mathcal{O}_W(9\Sigma) \otimes \mu^*(-6K)$ and $\mathcal{O}_C(\Sigma) \cong \mathcal{O}_C$. Thus we have proved the proposition in this case. Next we consider the case where $(C, \Sigma) = 1$. In the remainder we shall show that im J and im $H^0(\mathcal{O}_W(W)) \subset H^0(\mathcal{O}_C(W))$ generate $H^0(\mathcal{O}_C(W))$ in this case. Note that $C \subset H \subset W$, where H is the divisor in the assumption of the proposition. In this situation, $\{\ \partial F/\partial Z|_H = 0\ \}$ is a divisor on H

which has no intersections with $\Sigma|_H$. Here $\Sigma|_H$ denotes the restriction of Σ to H. Hence $\{ \partial F/\partial Z|_C = 0 \}$ consists of 18 points (which may contain multiple points.). Let V_1 denote the image of $H^0(\mathcal{O}_C(1))$ in $H^0(\mathcal{O}_C(W)) = H^0(\mathcal{O}_{\mathbb{P}^1}(21))$ under J. Then the linear system defined by V_1 consists of 18 fixed points and 3 points which move freely. In particular, $\dim V_1 = 4$. On the other hand, we have

$\{ \partial F/\partial Y|_H = 0 \} = 3\Sigma|_H +$ (effective divisors which have no intersections with $\Sigma|_H$).

Let V_2 denote the image of $H^0(\mathcal{O}_C(1) \otimes p^*K^{-3})$ in $H^0(\mathcal{O}_C(W))$ under J. Then $\dim V_2 = 10$. Finally let V_3 be the subspace of $\operatorname{im} H^0(\mathcal{O}_W(W)) \subset H^0(\mathcal{O}_C(W))$ which defines the linear system on C of the following type:

$9\Sigma|_C +$ (12 points which move freely).

We have $\dim V_3 = 13$. We shall consider the intersection of V_2 and V_3. $\{ \partial F/\partial Y|_C = 0 \}$ consists of 12 points; $3\Sigma|_C$(one point with multiplicity 3) and 9 points $R_1, \cdots, R_9$, none of which lies on Σ. Hence every section $s \in V_2 \subset H^0(\mathcal{O}_C(W))$ must be zero at these 12 points. On the other hand, we deduce from the definition of V_3 that $s^+ \in V_3 \subset H^0(\mathcal{O}_C(W))$ must be zero at $\Sigma|_C$ and that its multiplicity is at least 9. Therefore, we conclude that for every section $s \in V_2 \cap V_3$, $\{ p \in C ; s \text{ is zero at } p\} = 9\Sigma|_C + R_1 + \cdots + R_9 +$ others. This implies that $\dim(V_2 \cap V_3) \leq 4$. Since $\dim V_2 = 10$ and $\dim V_3 = 13$, we have $\dim(V_2 + V_3) \geq 19$. But every section of $V_2 + V_3$ must be zero at $\Sigma|_C$ with the multiplicity ≥ 3. Thus we obtain $\dim (V_2 + V_3) \leq 19$. Consequently, $\dim (V_2 + V_3) = 19$ and $\dim (V_2 \cap V_3) = 4$. Next we shall consider the intersection of V_1 and $(V_2 + V_3)$. Every section in V_1 is zero at $\{ \partial F/\partial Z|_C = 0\}$(which consist of 18 points different from $\Sigma|_C$). On the other hand, every section in $V_2 + V_3$ is zero at $\Sigma|_C$ with the multiplicity ≥ 3. Thus we conclude that $\dim V_1 \cap (V_2 + V_3) = 1$. From this we obtain

$$\begin{aligned}\dim(V_1 + V_2 + V_3) &= \dim V_1 + \dim(V_2 + V_3) - \dim(V_1 \cap (V_2 + V_3)) \\ &= 4 + 19 - 1 \\ &= 22.\end{aligned}$$

Since $\dim(V_1 + V_2 + V_3) = \dim H^0(\mathcal{O}_C(W)) = \dim H^0(\mathcal{O}_{\mathbb{P}^1}(21)) = 22$, $V_1 + V_2 + V_3$ coincides with $H^0(\mathcal{O}_C(W))$. This implies that im J and im $(H^0(\mathcal{O}_W(W)) \subset H^0(\mathcal{O}_C(W))$ generate $H^0(\mathcal{O}_C(W))$. Therefore, γ in (4.6) is surjective and the proof of the proposition is completed.

We are now in a position to state the following theorem.

Theorem. *Let W be a general nonsingular Weierstrass model over $\mathbb{P}^1 \times \mathbb{P}^1$. Then there are mutually disjoint rational curves $C_1, \cdots, C_4$ on W with the following properties:*

1) $N_{C_i/W} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ for each i.

2) $C_1, \cdots, C_4$ span $H_2(W\, ;\, \mathbb{Z})$.

Moreover, if we denote by $\varphi : W \longrightarrow \hat{W}$ the contraction of C_i's, then $\hat{W}$ can be deformed to a compact complex manifold W_t with $b_2(W_t) = 0$ and $K_{W_t} = \mathcal{O}_{W_t}$.

Proof.

By §3, there are mutually disjoint, isolated (-2)-curves C_i $(1 \leq i \leq 4)$ on W. Let C be one of C_i's. If C is a $(-2,c)$ – curve, then by (4.1) and (4.4), there is a suitable deformation of (C, W) in $\mathbb{P} = \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus K_{\mathbb{P}^1 \times \mathbb{P}^1}^{\otimes 2} \oplus K_{\mathbb{P}^1 \times \mathbb{P}^1}^{\otimes 3})$ such that C splits up into (-1,-1)-curves. Remark that a deformation of W in $\mathbb{P}$ is again a Weierstrass model over $\mathbb{P}^1 \times \mathbb{P}^1$. This is shown as follows. Consider the following diagram:

$$\begin{array}{ccc} \mathscr{W} & \supset & \mathscr{W}_0 = W \\ f\big\downarrow & & \big\downarrow \\ T & \supset & t_0 \ , \end{array}$$

where f is a flat deformation of W. We assume that T is a disk around t_0. Since $H^2(\mathcal{O}_W) = 0$, the line bundle $\mathcal{O}_W(\Sigma)$ is a restriction of a line bundle $\mathcal{L}$ on $\mathscr{W}$. Since $h^1(\mathcal{O}_W(\Sigma)) = h^3(\mathcal{O}_W(\Sigma)) = 0$ ($h^1 = 0$ follows from the spectral sequence of Leray for $\mu : W \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and $h^3 = 0$ follows from Serre duality), $h^0(\mathcal{L}_t)$ and $h^2(\mathcal{L}_t)$ are constant around t_0 by the invariance of $\chi(\mathcal{L}_t)$. In particular, $R^0 f_* \mathcal{L}$ is a locally free sheaf near t_0, which implies that a non-zero section of $\mathcal{L}_{t_0}$ extends to a non-zero section of $\mathcal{L}_t$ for a t near t_0. Hence Σ extends to a section Σ_t of μ_t: $\mathscr{W}_t \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. If μ_t has a reducible fibre, then some component F of this fibre does not intersect with Σ_t. This implies that $\rho(\mathscr{W}_t) \geq 4$ because Σ_t, $\mu_t^* \, p_1^* \mathcal{O}(1)$, $\mu_t^* \, p_2^* \mathcal{O}(1)$ and a divisor D with $(D. \ F) \neq 0$ span a subgroup of $NS(\mathscr{W}_t)$ of rank 4. But since $h^1(\mathcal{O}_W) = h^2(\mathcal{O}_W) = 0$, the Picard number is invariant under deformation, that is , $\rho(\mathscr{W}_t) = 3$. This is a contradiction. Therefore, $\mathscr{W}_t$ has only irreducible fibres and a section, which implies that $\mathscr{W}_t$ is a Weierstrass model over $\mathbb{P}^1 \times \mathbb{P}^1$.

By the above observation, it follows from Proposition(4.4) that there are mutually disjoint rational curves $C_1, \cdots, C_4$ with $N_{C_i/W} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ on a general Weierstrass model W over $\mathbb{P}^1 \times \mathbb{P}^1$. Then we have

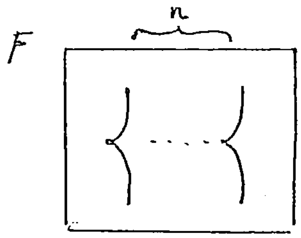
$$
(4.7)\quad
\begin{array}{lll}
(\Sigma\,,\ C_1) = 0 & (\mu^* p_1^* \mathcal{O}(1),\ C_1) = 1 & (\mu^* p_2^* \mathcal{O}(1),\ C_1) = 0 \\
(\Sigma\,,\ C_2) = 0 & (\mu^* p_1^* \mathcal{O}(1),\ C_2) = 0 & (\mu^* p_2^* \mathcal{O}(1),\ C_2) = 1 \\
(\Sigma\,,\ C_3) = 1 & (\mu^* p_1^* \mathcal{O}(1),\ C_3) = 0 & (\mu^* p_2^* \mathcal{O}(1),\ C_3) = 1 \\
(\Sigma\,,\ C_4) = 1 & (\mu^* p_1^* \mathcal{O}(1),\ C_4) = 1 & (\mu^* p_2^* \mathcal{O}(1),\ C_4) = 0.
\end{array}
$$

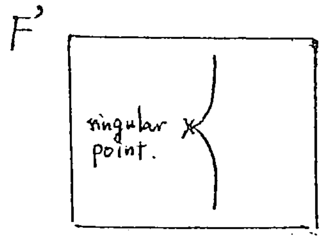
Hence $C_1, \cdots, C_4$ generate $H_2(W\,;\,\mathbb{Z})$ by Appendix. Moreover using (4.7) we can show that for $\widehat{W}$ the assumption of Corollary(4.7) in [2] is satisfied. Therefore $\widehat{W}$ is smoothable. Q.E.D.

Appendix. the calculation of $H_2(W\ ;\ \mathbb{Z})$

Let $\mu: W \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be a nonsingular Weierstrass model. We shall show that $H_2(W\ ;\ \mathbb{Z})$ is torsion free and make explicit the generators of $H_2(W\ ;\ \mathbb{Z})$.

a) Let $(T_0 : T_1) \times (S_0 : S_1)$ be the homogenous coordinate of $\mathbb{P}^1 \times \mathbb{P}^1$. Set $n = 12$ and $b = T_1^n S_1^n + T_0^n S_1^n + T_1^n S_0^n + 2T_0^n S_0^n$. Then $b = 0$ defines a nonsingular curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$. Consider the Weierstrass model W defined by $Y^2Z = X^3 + bZ^3$, which is nonsingular. Let p_1 be the first projection of $\mathbb{P}^1 \times \mathbb{P}^1$. Then C is a n-cover of $\mathbb{P}^1$, and there are 2n ramification points $\{P_1, \ldots, P_{2n}\} \in C$ for which the ramification index $\delta(P_i) = n$ $(1 \leq i \leq 2n)$. Put $\sigma = p_1 \circ \mu$. Then σ is a $K3$ fibration with n singular fibres. A general (resp. singular) fibre F (resp. F') has an elliptic fibration α for which the singular fibres are $n \times \mathrm{II}$ (resp. $1 \times \mathrm{II}$).





b) Let F be a general fibre of σ. Then we have

$$H_1(F\ ;\ \mathbb{Z}) = 0$$

$$H_2(F\ ;\ \mathbb{Z}) = \mathbb{Z}e \oplus \mathbb{Z}f_1 \oplus \ldots \oplus \mathbb{Z}f_{2(n-2)} \oplus \mathbb{Z}g \quad ,$$

where the generators e and g are represented by the following 2-cycles on F.

e := a smooth fibre of $\alpha: F \longrightarrow \mathbb{P}^1$, which is homeomorphic to $S^1 \times S^1$

g := the canonical section $\Sigma|_F$ of α, which is homeomorphic to S^2

$\mathbb{Z}f_1 \oplus \ldots \oplus \mathbb{Z}f_{2(n-2)}$ is defined by two exact sequences:

$$(1)\quad 0 \longrightarrow Ker \longrightarrow \bigoplus_{i=1}^{n} H_1(S^1 \times S^1;\ \mathbb{Z}) \longrightarrow H_1(S^1 \times S^1;\ \mathbb{Z})$$

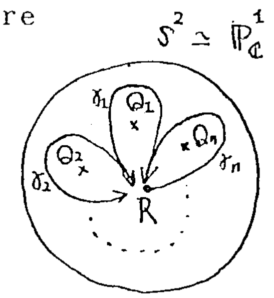
$$(a_1, \dots, a_n) \longrightarrow \sum_{i=1}^{n} a_i$$

$$(2)\quad 0 \longrightarrow \mathbb{Z}^{\oplus 2} \longrightarrow Ker \longrightarrow \mathbb{Z}f_1 \oplus \dots \oplus \mathbb{Z}f_{2(n-2)} \longrightarrow 0$$

$$a = {}^t(a_1,\ a_2) \longrightarrow (a,\ Aa, \dots,\ A^{n-1}a)$$

A: monodoromy matrix for a singular fibre

Figure



The fibration $\alpha : F \longrightarrow \mathbb{P}^1_{\mathbb{C}} \simeq S^2$ has singular fibres over $\{Q_1, \cdots, Q_n\}$.

$\gamma_i : [0,1] \longrightarrow S^2$ is a circle around Q_i with $\gamma_i(0) = R$, $\gamma_i(1) = R$.

In the exact sequence (1), an element $(a_1, \dots, a_n) \in \bigoplus_{i=1}^{n} H_1(S^1 \times S^1;\ \mathbb{Z})$ corresponds to a 2-chain $C_1 + \dots + C_n$; $C_i = \{C_i(t)\}_{t \in [0,1]}$, $a_i = C_i(1) - C_i(0)$, where $\{C_i(t)\}$ is a 1-cycle in $S^1 \times S^1 \simeq \alpha^{-1}(\gamma_i(t))$. Therefore an element in *Ker* is a 2-chain which goes to zero by the boundary map ∂ .

$\mathbb{Z}f_1 \oplus \dots \oplus \mathbb{Z}f_{2(n-2)}$ is the quotient of *Ker* by the equivalence relation $\sim$; $(b_1, \dots, b_n) \sim (b_1^+, \dots, b_n^+) \leftrightarrow (b_1, \dots, b_n) = (b_1^+, \dots, b_n^+) + (a,\ Aa, \dots,\ A^{n-1}a)$ for $a \in H_1(S^1 \times S^1\ ;\ \mathbb{Z})$.

Let γ be a circle around $P_i \in \mathbb{P}^1$. Then the monodoromy action γ_* for the singular fibre $\sigma^{-1}(P_i)$ is described as follows.

(3)

$$\gamma_*|_{\mathbb{Z}e} = \mathrm{id} : \mathbb{Z}e \longrightarrow \mathbb{Z}e$$

$$\gamma_*|_{\mathbb{Z}g} = \mathrm{id} : \mathbb{Z}g \longrightarrow \mathbb{Z}g$$

$$\begin{array}{ccc}
(a_1, \dots, a_{n-1}, a_n) \in Ker & \longrightarrow & \mathbb{Z}f_1 \oplus \dots \oplus \mathbb{Z}f_{2(n-2)} \\
\downarrow \qquad\qquad \downarrow & & \downarrow \gamma_* \\
(Aa_n, Aa_{n-1}, \dots, Aa_1) \in Ker & \longrightarrow & \mathbb{Z}f_1 \oplus \dots \oplus \mathbb{Z}f_{2(n-2)}
\end{array}$$

c) Let F' be a singular fibre of σ. Then we have

$$H_1(F';\mathbb{Z}) = 0$$
$$H_2(F';\mathbb{Z}) = \mathbb{Z}e \oplus \mathbb{Z}g \quad ,$$

where

e := a smooth fibre of α: $F' \longrightarrow \mathbb{P}^1$, which is homeomorphic to $S^1 \times S^1$

g := the canonical section $\Sigma|_{F'}$ of α, which is homeomorphic to S^2

d) Consider σ: $W \longrightarrow \mathbb{P}^1(\mathbb{C}) \simeq S^2$. S^2 is coverd by two open disks U and V with the following properties: $P_i \in U$ $(1 \leq i \leq 2n-1)$, $P_{2n} \in U$ $P_{2n} \in V$, $P_i \in V$ $(1 \leq i \leq 2n-1)$ and $U \cap V \underset{\text{homeo}}{\simeq} S^1 \times (0,1)$. Moreover U is coverd by $(2n - 1)$ open sets U_i $(1 \leq i \leq 2n - 1)$ with the following properties: I) $P_j \in U_i$ if and only if i = j. II) For i and j with $i + 1 \leq j$, $U_i \cap U_j \neq \phi$ if and only if $j = i + 1$. III) $U_i \cap U_j$ is contractible to a point. By Mayer Vietoris sequence for $\sigma^{-1}(U_i)$'s, we have

$$H_1(\sigma^{-1}(U)\ ;\mathbb{Z}) = 0 \quad \text{and} \quad H_2(\sigma^{-1}(U)\ ;\mathbb{Z}) = \mathbb{Z}e \oplus \mathbb{Z}g \ ,$$

where we use the same notation in b). Obviously,

$$H_1(\sigma^{-1}(V)\ ;\mathbb{Z}) = 0 \quad \text{and} \quad H_2(\sigma^{-1}(V)\ ;\mathbb{Z}) = \mathbb{Z}e \oplus \mathbb{Z}g \ .$$

Again by the Mayer Vietoris sequence for $\sigma^{-1}(U)$ and $\sigma^{-1}(V)$, we have

$$\begin{array}{l} H_2(\sigma^{-1}(U) \cap \sigma^{-1}(V)\ ;\mathbb{Z}) \longrightarrow H_2(\sigma^{-1}(U)\ ;\mathbb{Z}) \oplus H_2(\sigma^{-1}(V)\ ;\mathbb{Z}) \\ \longrightarrow H_2(W\ ;\mathbb{Z}) \longrightarrow \underset{\displaystyle \overset{||}{\mathbb{Z}}}{H_1(\sigma^{-1}(U) \cap \sigma^{-1}(V)\ ;\mathbb{Z})} \longrightarrow 0 \ . \end{array}$$

In the above sequence, $H_2(\sigma^{-1}(U) \cap \sigma^{-1}(V)\ ;\mathbb{Z}) \simeq \mathbb{Z}e \oplus \mathbb{Z}g \oplus$ (torsion) by (3) in b), which implies that $H_2(W\ ;\mathbb{Z}) \simeq \mathbb{Z}^{\oplus 3}$. Since $\Sigma \simeq \mathbb{P}^1 \times \mathbb{P}^1$, Σ has two rulings, one of which coincides with g. We denote by h another one. Then $H_2(W\ ;\mathbb{Z}) \simeq \mathbb{Z}e \oplus \mathbb{Z}g \oplus \mathbb{Z}h$, where e is a fibre of

$\mu\colon W \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$

References

1. Clemens, H.: Homological equivalence modulo algebraic equivalence is not finitely generated, Publ. Math. I.H.E.S **58**. 19 ~ 38 (1983)
2. Friedman, R.: Simultaneous resolution of threefold double points, Math. Ann. **274**. 671 ~ 689 (1986)
3. Grothendieck, A.: Technique de construction et Theoremes d'existence en geometie algebraique IV: Le schemas de Hilbert exposes 221, Secretariat mathematique, 11 rue Pierre Curie, Paris 5e (1961)
4. Kawamata, Y.: The crepant blowing-ups of 3-dimensional canonical singularities and its applications to degeneration of surfaces Ann. of Math. **127**. 93 ~ 163 (1988)
5. Kodaira, K. & Spencer, D.C.: A theorem of completeness for complex analytic fibre spaces Acta. Math. **100**. 281 ~ 294 (1958)
6. Nakayama, N.: On Weierstrass models, in Algebraic Geomety and Commutative Algebra vol.II in honor of M. Nagata, Kinokuniya (1989)
7. Namikawa, Y.: On the birational structure of some Calabi-Yau threefold in preparation. Kyoto Univ.(1989)
8. Reid, M.: Minimal models of canonical 3-folds in Adv. Studies of Pure Math. I, Algebraic Varieties and Analytic Varieties, Proceedings of a symposium held in Tokyo, 1981 edited by Iitaka Kinokuniya, North-Holland 131 ~ 180 (1983)
9. Reid, M.: The moduli space of 3-fold with $K = 0$ may nevertheless be irreducible Math. Ann. **278**. 329 ~ 334

10. Schoen, C.: On fibre products of rational elliptic surfaces with sections Math. Z. **197**. 177 ~ 199 (1988)

11. Wall, C.T.C.: Classification problem in topology V : On certain 6-manifolds Invent. Math. **1**. 355 ~ 374 (1966)

12. Barth, Peters, Van de Ven.: Compact complex surfaces Springer-Verlag